Temperature Equilibration of a 1D Coulomb Chain and a Many-Particle Adiabatic Invariant

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Several recent experiments have produced an ordered chain of ions which are confined along a one-dimensional axis. Here we examine the rate of irreversible energy transfer between degrees of freedom describing motions transverse and parallel to this axis. Because of the strong transverse confinement, the transverse motions are much higher frequency than the parallel motion, and so a many-particle adiabatic invariant exists which greatly reduces the rate of thermal equilibration.

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The one-dimensional Coulomb chain is a simple form of condensed matter consisting of charges of a single species trapped in a linear configuration through the application of strong external magnetic and/or electric fields. Recently, such chains have been realized in two experiments [1,2], in which the charges have been cooled into the regime of strong correlation where the correlation parameter $\Gamma \equiv q^2/aT$ is much larger than unity. (Here q is the ion charge, T is the temperature, and a is the average intercharge spacing.) The 1D chain has been suggested as an advantageous configuration for a novel type of atomic clock based on trapped ions [2,3]. It has also been predicted that such chains may form in heavy ion storage rings provided that sufficiently strong electron or laser cooling is applied [4]. Such cold 1D chains would provide an attractive low emittance ion source.

Although the charges are strongly bound to the axis of the trap or the storage ring by the applied forces, high frequency transverse motions still occur and the temperature $T_\perp$ associated with these motions need not be the same as that associated with the motions parallel to the axis, $T_\parallel$. For example, when laser cooling or electron cooling is applied along the chain axis, the transverse oscillations are not directly cooled and come to equilibrium with the parallel motion only indirectly through Coulomb collisions [5]. In this case the overall cooling rate depends on the rate at which collisions cause $T_\perp$ and $T_\parallel$ to equilibrate. This equilibration rate has been examined via numerical simulations [6]. However, the regime in which both parallel and transverse motions are of small amplitude (near harmonic) has not yet been explored, and it is often in this regime that the experiments operate.

In this paper we calculate the rate ν at which an anisotropic temperature distribution relaxes to thermal equilibrium in a strongly correlated ($\Gamma \gg 1$) 1D chain in the strongly focusing limit, where the motions transverse to the axis are of high frequency compared to the parallel motions. Because of this time-scale separation we find that a many-particle adiabatic invariant exists equal to the total action associated with the transverse motions. If this approximate invariant were exactly conserved, equilibration could not occur. However, we find that N-body collisions cause small changes in the invariant, leading to a slow rate of equilibration, exponentially small in the ratio of transverse to parallel frequencies.

Our model for the trap consists of a harmonic radial confining potential of the form $m\omega_r^2(x^2+y^2)/2$ where $\mathbf{r}=(x,y,z)$ are Cartesian coordinates with z oriented along the beam axis. In the strong focusing limit of interest here, the parameter $\varepsilon \equiv \omega_0/\omega_r$ is small, where $\omega_0 \equiv \sqrt{q^2/ma^3}$ is a plasma frequency associated with parallel oscillations. This radial potential is an excellent approximation for the linear [2] and circular [1] Paul trap experiments, and is a useful first approximation for the comoving frame of ions in a storage ring [4]. The Hamiltonian for the N-ion system is then written as

$$H(\mathbf{r}_1,\mathbf{p}_1,\dots\mathbf{r}_N,\mathbf{p}_N)=\sum_{n=1}^{N}(p_n^2/2m+m\omega_r^2[x_n^2+y_n^2]/2)+\sum_{l>n}q^2/\sqrt{x_{ln}^2+y_{ln}^2+[z_{ln}+a(l-n)]^2},$$

where $\mathbf{r}_{ln} \equiv \mathbf{r}_l - \mathbf{r}_n$, and for each ion $\mathbf{r}_n$ is measured from its equilibrium position in the linear chain. For simplicity we assume here that in equilibrium the ions are equally spaced, as in the ring trap, and image charges and curvature effects, if any, are neglected.

When the ions are strongly correlated, the dynamics is dominated by N-body processes rather than two-body collisions. Here we assume that both $\Gamma_\perp \equiv q^2/aT_\perp$ and $\Gamma_\parallel \equiv q^2/aT_\parallel$ are sufficiently large so that we may describe the ion-ion interaction as emission and absorption of phonons. The ideal phonon limit is then attained by expansion of the Coulomb potential in $|\mathbf{r}_n|/a$ to second order in this small quantity. The resulting harmonic Hamiltonian H_0 then describes N eigenmodes with polarizations parallel to z and $2N$ transverse modes. The parallel and transverse mode frequencies are given by

$$\omega_z(k)=\omega_0[8\sum_{n=1}^{\infty}\sin^2(nk/2)/n^3]^{1/2}$$

and $\omega_\perp(k)=\sqrt{\omega_r^2-\omega_z^2(k)/2}$, respectively, where $k=2\pi n/N$ ($n=0,1,\dots,N-1$) is the parallel wave vector of the eigenmodes normalized to a [6].

Even at low temperatures, anharmonic terms neglected

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in H_0 but present in H couple the parallel (or transverse) phonons to one another, e.g., through three phonon collisions. This low-order phonon-phonon coupling is expected to cause the distribution of parallel (or transverse) energy to relax to a Maxwellian described by a temperature $T_\parallel$ ($T_\perp$). However, when $\varepsilon \ll 1$ energy conservation does not allow these low-order processes to create or destroy transverse phonons, because annihilation of a single transverse phonon requires creation of many parallel phonons.

The total number of quanta (i.e., the total action) associated with the high frequency transverse motions is then an adiabatic invariant. In order for the transverse and parallel temperatures to equilibrate this invariant must be broken: Transverse phonons must be created or annihilated. In fact, the symmetry of H in x and y implies that transverse phonons must be created or destroyed in pairs. The rate ν for parallel to transverse equilibration can then be estimated using an order of magnitude estimate based on Fermi's golden rule: $\nu \sim \omega_0 \langle (\Delta H/H_0)^2 \rangle$ where ΔH is the interaction energy for a process which annihilates two transverse phonons, and $\langle\ \rangle$ denote a statistical average. Recognizing that about M parallel phonons must be created in this process, where $M = 2\omega_r/\omega_m$ and $\omega_m = \sqrt{7\zeta(3)}\omega_0$ is the maximum parallel phonon frequency, we crudely approximate ΔH as a Taylor expansion of H: $\Delta H/H_0 \sim z^M(x^2+y^2)/a^{M+2}$. We perform the average using a harmonic Einstein approximation for the distribution of displacements, proportional to $\exp(-[\Gamma_\parallel z^2 + \Gamma_\perp(x^2+y^2)/\varepsilon^2]/a^2)$. Neglecting an unimportant multiplicative constant, the average yields

$$\nu \sim (\omega_0 \varepsilon^4/\Gamma_\perp^2)\exp(-2\{1+\ln[\sqrt{7\zeta(3)}\varepsilon\Gamma_\parallel/2]\}/\sqrt{7\zeta(3)}\varepsilon), \tag{1}$$

which is exponentially small, as expected. Note, however, that $\varepsilon\Gamma_\parallel$ must be greater than unity in order for the result to be sensible, because the average is dominated by z displacements with a peak at $z/a \sim 1/\sqrt{\varepsilon\Gamma_\parallel}$. That is, large displacements in z would make a large contribution to the rate, but such displacements are improbable. When $\varepsilon\Gamma_\parallel > 1$ small displacements make the main contribution to ν, consistent with the assumption of harmonic fluctuations.

To calculate the equipartition rate more rigorously, we perform a series of three canonical transformations in order to isolate the total transverse action variable J_0. We first transform to phonon coordinates $(\tilde{\mathbf{r}}_k, \tilde{\mathbf{p}}_k)$, through the Fourier relations

$$(\tilde{\mathbf{r}}_k, \tilde{\mathbf{p}}_k) = N^{-1/2}\sum_{l=0}^{N-1}(\mathbf{r}_l e^{-ikl}, \mathbf{p}_l e^{ikl}) .$$

In these coordinates H_0 has the form of $3N$ uncoupled harmonic oscillators of frequencies ω_j, where j refers to both wave number k and polarization direction ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$): $H_0 = \sum_j [p_j^2/2m + m\omega_j^2 r_j^2/2]$. We next transform the $2N$ transverse phonon variables to $2N$ action angle pairs (ψ_j, I_j) via the transformation $(\tilde{r}_j, \tilde{p}_j) = \sqrt{2I_j/m\omega_j} \times(\sin\psi_j, m\omega_j\cos\psi_j)$. The angle variables ψ_j evolve on a time scale of order ω_r^{-1}. Finally, we apply the canonical transformation [7] $\theta_0 = \psi_0$, $\theta_j = \psi_j - \psi_0$ ($j \neq 0$), $J_0 = \sum_j I_j$, $J_j = I_j$ ($j \neq 0$). Now only θ_0 varies at ω_r^{-1}; all other variables are slowly varying. The total transverse action J_0 is therefore an adiabatic invariant.

We are interested in the time rate of change of J_0 averaged over a suitably chosen statistical distribution D of systems, $d\langle J_0\rangle/dt = \int d\Lambda\, D[J_0, H]_\Lambda$, where Λ is a point in the $6N$-dimensional phase space, and $[\cdot,\cdot]_\Lambda$ is a Poisson bracket. At some time in the past, long compared to the relaxation time to a two-temperature Maxwellian but short compared to the $T_\perp \to T_\parallel$ relaxation time, we assume that D was a two-temperature Maxwellian, written as $D_0 = Z^{-1}\exp\{-\omega_r J_0/T_\perp - (H - \omega_r J_0)/T_\parallel\}$. However, since J_0 is not an exact constant of the motion, a fluctuation D_1 develops which may be obtained through solution of Liouville's equation with D_0 as the initial condition, $D_1(\Lambda, t) = -\int_{-\infty}^{t} dt' [D_0, H]_{\Lambda(t')}$, where the Poisson bracket is evaluated along the phase-space trajectory $\Lambda(t')$ for which $\Lambda(t) = \Lambda$, and where the slow time dependence of $T_\perp$ and $T_\parallel$ has been neglected. Substitution of D_1 into $d\langle J_0\rangle/dt$ then yields

$$d\langle J_0\rangle/dt = \int d\Lambda\, D_0[J_0, H]_\Lambda + (1/T_\perp - 1/T_\parallel)(2\varepsilon)^{-1}\int_{-\infty}^{+\infty} d\tau\, C(\tau) , \tag{2}$$

where we introduce the correlation function $C(\tau) \equiv \langle \dot{J}_0(t)\dot{J}_0(0)\rangle$, $\tau \equiv \omega_0 t$, $\dot{J}_0 = -\partial H/\partial\theta_0$, and where $\langle\ \rangle$ represents an average over D_0. The first term of Eq. (2) vanishes because D_0 depends on Λ only through J_0 and H; and the time integral in the second term has been extended to $t = +\infty$ using the symmetry $C(\tau) = C(-\tau)$.

However, the time integral in Eq. (2) cannot be evaluated because it involves the exact trajectory $\Lambda(t)$. We follow standard practice [8] by substituting approximate trajectories $\Lambda^{(0)}(t)$, in this case determined by the harmonic Hamiltonian H_0, and we also replace $D_0(H, J_0)$ by $D_0(H_0, J_0)$. That is, we approximate the dynamics by that of an ideal phonon gas, so the system must be strongly correlated, i.e., $\Gamma_\parallel \gg 1$. We also assume here that the parallel force due to transverse displacements, of order $q^2 r_{ln}^2/a^4$, can be treated as a small perturbation of the parallel motion, which requires $\Gamma_\perp \gg \varepsilon^2\sqrt{\Gamma_\parallel}$. The substitution of $\Lambda(t)$ by $\Lambda^{(0)}(t)$ is a major assumption of our calculation. Despite the fact that this type of assumption works well for a weakly correlated plasma [7], its validity needs to be tested for a strongly correlated plasma. Furthermore, we expect that processes involving creation and annihilation of only two transverse phonons will dominate the equilibration rate so we Taylor expand $\partial H/\partial\theta_0$ in x_{ln}^2 and y_{ln}^2 keeping only lowest order nonzero terms, $\dot{J}_0 = (q^2/2)\sum_{l>n} Z_{ln}^{-3}\,\partial r_{ln}^2/\partial\theta_0$, where $r_{ln}^2 \equiv x_{ln}^2 + y_{ln}^2$ and $Z_{ln} \equiv (l-n)a + z_{ln}$ is the z distance between ions l and n.

With these assumptions we find that the averages over transverse and parallel phonons appearing in $C(\tau)$ decouple: $C(\tau) = T_\perp^2 \sum_{\mathbf{m}} C_{\mathbf{m}}^{\perp}(\tau, \varepsilon) C_{\mathbf{m}}^{\parallel}(\tau, \Gamma_\parallel)$, where $\mathbf{m} \equiv (l, n, \bar{l}, \bar{n})$, and the sum runs over all $l > n$, $\bar{l} > \bar{n}$. The (dimensionless) parallel and transverse parts of $C(\tau)$ are $C_{\mathbf{m}}^{\parallel}(\tau,\Gamma_\parallel)\equiv 4a^6\langle Z_{ln}^{-3}(\tau)Z_{ln}^{-3}(0)\rangle$ and $C_{\mathbf{m}}^{\perp}(\tau,\varepsilon)\equiv\Gamma_\perp^2\langle[\partial r_{ln}^2(\tau)/\partial\theta_0]\partial r_{ln}^2(0)/\partial\theta_0\rangle/16a^4$, respectively. Employing harmonic phonon orbits $\Lambda^{(0)}(t)$ to determine $r_{ln}^2(\tau)$, the average in $C_{\mathbf{m}}^{\perp}$ can be performed explicitly:

$$C_{\mathbf{m}}^{\perp}(\tau,\varepsilon)=2\varepsilon^4(\{[S_{\mathbf{m}}^{+}(\varepsilon\tau)]^2-[S_{\mathbf{m}}^{-}(\varepsilon\tau)]^2\}\cos(2\tau/\varepsilon)+2S_{\mathbf{m}}^{+}(\varepsilon\tau)S_{\mathbf{m}}^{-}(\varepsilon\tau)\sin(2\tau/\varepsilon))\,,$$

where the functions $S_{\mathbf{m}}^{+}$ and $S_{\mathbf{m}}^{-}$ are defined as

$$S_{\mathbf{m}}^{(+)}(\tau)\equiv\int_0^\pi\frac{dk}{2\pi}[\cos k(l-\bar{l})+\cos k(n-\bar{n})-(l\leftrightarrow n)]\cos[\omega_\parallel^2(k)\tau/4\omega_0^2]\,,$$

$$S_{\mathbf{m}}^{(-)}(\tau)\equiv\int_0^\pi\frac{dk}{2\pi}[\cos k(l-\bar{l})+\cos k(n-\bar{n})-(l\leftrightarrow n)]\sin[\omega_\parallel^2(k)\tau/4\omega_0^2]\,.$$

Similarly, the use of harmonic phonons in the parallel average implies $C_{\mathbf{m}}^{\parallel}=\int_0^\beta dx_1dx_2 g_{\mathbf{m}}(x_1,x_2)h_{\mathbf{m}}(\tau,2x_1x_2/\Gamma_\parallel)$ where the functions $g_{\mathbf{m}}$ and $h_{\mathbf{m}}$ are given by

$$g_{\mathbf{m}}(x_1,x_2)=(x_1x_2)^2\exp\{-x_1(l-n)-x_2(\bar{l}-\bar{n})+2[x_1^2f_{l-n}(0)+x_2^2f_{\bar{l}-\bar{n}}(0)]/\Gamma_\parallel\}\,,$$
$$h_{\mathbf{m}}(\tau,\alpha)=\exp\{-\alpha[f_{l-\bar{l}}(\tau)+f_{n-\bar{n}}(\tau)-(l\leftrightarrow n)]\}\,, \tag{3}$$

and where the time dependence enters only through the function $f_{l-n}(\tau)\equiv\Gamma_\parallel\langle z_{ln}(\tau)z_{ln}(0)\rangle/4a^2$ which can be written in terms of the parallel phonon spectrum:

$$f_n(\tau)=\int_0^\pi dk\frac{1-\cos kn}{2\pi\omega_\parallel^2(k)/\omega_0^2}\cos[\omega_\parallel(k)\tau/\omega_0]\,.$$

In deriving this expression for $C_{\mathbf{m}}^{\parallel}$ we have replaced Z_{ln}^{-3} by a smoothed function dependent on a parameter β: $Z_{ln}^{-3}\simeq\int_0^\beta dx_1x_1^2\exp(-x_1Z_{ln}/a)/2a^3$. This is exact for $\beta\to\infty$, but for finite β it avoids the singularity in Z_{ln}^{-3} which occurs for close collisions, i.e., when $Z_{ln}\to 0$. This singularity is disallowed under exact dynamics, but is allowed in the harmonic dynamics which we employ, and it would lead to a singular result for $C_{\mathbf{m}}^{\parallel}$ [this can be observed in $g_{\mathbf{m}}(x_1,x_2)$, which blows up as x_1 or x_2 approaches ∞]. However, we will find that a range of large but finite β values exist for which $C_{\mathbf{m}}^{\parallel}$ is independent of β, provided that $\varepsilon\Gamma_\parallel\gg 1$. Only then is $C_{\mathbf{m}}^{\parallel}$ dominated by small z displacements, just as in Eq. (1).

To evaluate Eq. (2) we first perform the time integral $I_{\mathbf{m}}(\varepsilon,\alpha)\equiv\int_{-\infty}^{+\infty}d\tau\, C_{\mathbf{m}}^{\perp}(\tau,\varepsilon)h_{\mathbf{m}}(\tau,\alpha)$. The function $h_{\mathbf{m}}$, associated with parallel fluctuations, is slowly varying compared to the rapid oscillations of $C_{\mathbf{m}}^{\perp}$; this leads to an exponentially small result for $I_{\mathbf{m}}$. It is also important to note that $C_{\mathbf{m}}^{\perp}\to 0$ on a time scale of order $(\varepsilon\omega_0)^{-1}$ due to phase mixing of the transverse phonons; that is, S^+ and $S^-\to 0$ on this time scale, so the integral is convergent. We evaluate $I_{\mathbf{m}}$ using the saddle-point method in the complex τ plane. Since the integrand is an entire function of τ, we can deform the contour through the saddle points. Their positions depend on $\mathbf{m}$ but we have found that the integral obtained from nearest neighbor interactions, $\mathbf{m}^*(=l,l-1,l,l-1)$, dominates the final result for ν so we keep only this term. The saddle-point positions are then solutions of the saddle-point equation $\dot{f}_1(\tau)=-i/\varepsilon\alpha$. Because $f_1(\mathrm{Re}[\tau])$ is oscillatory, there are an infinite number of solutions distributed symmetrically on each side of the imaginary τ axis, as well as one pure imaginary solution. However, for small ε only a few saddle points nearest $\mathrm{Re}[\tau]=0$ need to be kept, and in fact the pure imaginary saddle point gives the main trend of the integral. A comparison of the saddle-point method and direct numerical integration is shown in Fig. 1.

An important feature of Fig. 1 is the abrupt steps in $I_{\mathbf{m}^*}$ at integer ratios between frequencies $2\omega_r$ and the maximum parallel phonon frequency $\omega_m=\sqrt{7\zeta(3)}\omega_0$. These steps are a consequence of the fact that the frequency spectrum of the harmonic parallel dynamics [described by $f_1(\tau)$] exhibits a sharp cutoff at ω_m. This implies that a phonon-phonon interaction which creates or annihilates two transverse phonons and M parallel phonons can only occur if $M\omega_m>2\omega_r$, or $\varepsilon^{-1}<M\sqrt{7\zeta(3)}/2$. When ε^{-1} exceeds this value the process no longer contributes and the rate decreases abruptly. For very large

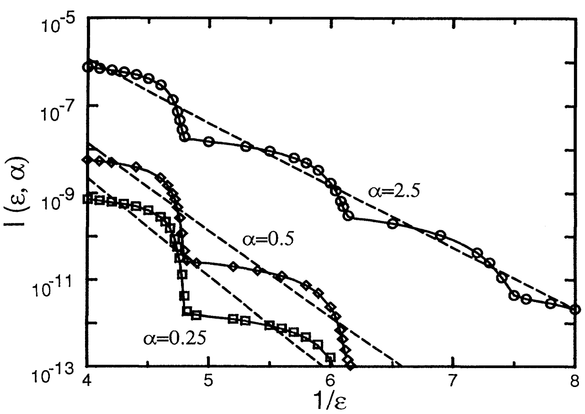

FIG. 1. Plot of the time integral $I_{\mathbf{m}^*}(\varepsilon,\alpha)$ for different α values. Solid lines: saddle-point calculation keeping eleven saddle points on each side of the imaginary τ axis and the pure imaginary saddle point. Dashed lines: saddle-point calculation keeping only the pure imaginary saddle point. Symbols: direct numerical integration; ○: $\alpha=2.5$, ◇: $\alpha=0.5$, and □: $\alpha=0.25$.

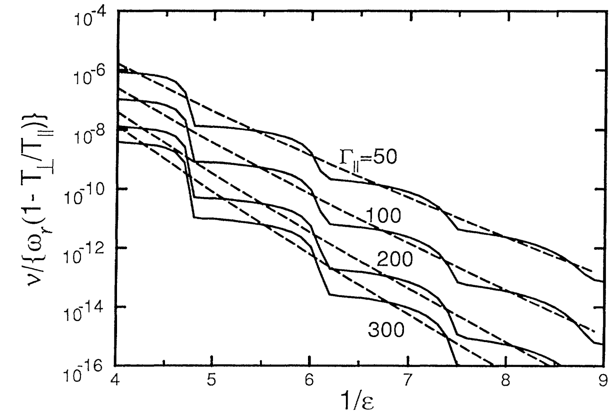

FIG. 2. Plot of $\bar{\nu}(\varepsilon,\Gamma_\|)=\nu/\{\omega_r(1-T_\perp/T_\|)\}$ for different values of $\Gamma_\| \equiv q^2/aT_\|$. Here ν is the equilibration rate, and $\varepsilon \equiv \omega_0/\omega_r$. The dashed lines represent Eq. (5). Equation (5) becomes a better approximation for larger $1/\varepsilon$ and smaller $\Gamma_\|$.

ε^{-1} these steps are smoothed out and finally disappear because the rate is then determined by many high-order processes, each of which has a small effect when taken individually.

To complete the rate calculation we evaluate the integral

$$\bar{\nu}(\varepsilon,\Gamma_\|) \equiv (4\varepsilon)^{-1} \int_0^\beta dx_1 dx_2 g_{\mathbf{m}^*}(x_1,x_2) \times I_{\mathbf{m}^*}(\varepsilon, 2x_1x_2/\Gamma_\|) . \quad (4)$$

The integral is performed by direct numerical integration. The equilibration rate $\nu \equiv \dot{T}_\perp/T_\perp$ can be written as $\nu=\omega_r(1-T_\perp/T_\|)\bar{\nu}(\varepsilon,\Gamma_\|)$ where the approximation $\langle J_0 \rangle \simeq 2NkT_\perp/\omega_r$ has been employed. The integrand in Eq. (4) is sharply peaked near $x_1,x_2 \sim 1/\varepsilon$, but begins to diverge at large x_1 and x_2 due to the aforementioned unphysical singularity in Z_{ln}^{-3}. However, we find that the integral is independent of β provided that we choose $1/\varepsilon \ll \beta \lesssim \Gamma_\|$, which implies $\varepsilon\Gamma_\| \gg 1$. Only under this condition will the harmonic phonon approximation be valid.

The scaled equilibration rate $\bar{\nu}(\varepsilon,\Gamma_\|)$ is shown in Fig. 2. The rate is strongly reduced as ε decreases. As we have discussed, the rather striking steps in the rate stem from the existence of a maximum frequency in the parallel dynamics, and are a qualitative signature of the strongly correlated regime. Such steps do not occur in weakly correlated plasma where binary interactions dominate and no sharp frequency cutoff exists in the relative parallel dynamics, Indeed, Fig. 2 shows that the steps decrease in magnitude as $\Gamma_\|$ decreases.

The dashed line in Fig. 2 is the result for $\bar{\nu}(\varepsilon,\Gamma_\|)$ when only the single pure imaginary saddle point is kept in $I_{\mathbf{m}^*}$. In this case a saddle-point evaluation of the integrals in Eq. (4) yields

$$\bar{\nu}(\varepsilon,\Gamma_\|)=\sqrt{\pi^7\varepsilon/8\eta^2}(\alpha_0\Gamma_\|)^{5/2}[S^+_{\mathbf{m}^*}(i\varepsilon\tau_0)+S^-_{\mathbf{m}^*}(i\varepsilon\tau_0)]^2 \exp[-2\tau_0/\varepsilon+2\alpha_0 f_1(i\tau_0)-\sqrt{2\Gamma_\|\alpha_0}] , \quad (5)$$

where $\alpha_0=(7/8+1/\eta\varepsilon)^2/\Gamma_\|$, $i\tau_0$ is the pure imaginary solution of the saddle-point equation evaluated at $\alpha=\alpha_0$, and $\eta \equiv \sqrt{7\zeta(3)}$. When $\varepsilon\Gamma_\| \gg 1$, $\tau_0 \simeq \gamma+(\ln\gamma)/2\eta$, where $\gamma \equiv \ln[\sqrt{\pi}\eta \ln 2/\varepsilon\alpha_0]/\eta$. As either ε or $\Gamma_\|$ decreases, Eq. (5) becomes a better approximation to $\bar{\nu}(\varepsilon,\Gamma_\|)$ (see Fig. 2). To lowest order in ε and $(\varepsilon\Gamma_\|)^{-1}$ the exponential dependence in Eq. (5), $\exp[-2\ln(\varepsilon\Gamma_\|)/\eta\varepsilon]$, is the same as the crude estimate of Eq. (1).

In order for our calculation to be valid, the aforementioned conditions $\varepsilon \ll 1$, $\varepsilon\Gamma_\| \gg 1$, and $\Gamma_\perp \gg \varepsilon^2\sqrt{\Gamma_\|}$ must be satisfied. In fact, these conditions are not fully satisfied in the previous molecular dynamics calculation [6] (the last condition in particular) and therefore a detailed comparison between that calculation and the present analysis is not possible. However, Ref. [6] does document a decrease in the equilibration rate as ε decreases. New simulations are underway in order to test our results. We also note that other mechanisms, such as scattering with gas molecules or heating due to the rf micromotion in the trap, may contribute to the equilibration process in a real Paul trap or storage ring.

Finally, it is worth noting that there is a strong similarity between the present problem and the perpendicular to parallel temperature equilibration of a crystallized single species plasma in the strong magnetization limit, where the cyclotron frequency is large compared with the plasma frequency; now the cyclotron frequency assumes the role of ω_r. This equilibration process has been examined by O'Neil and Hjorth for a weakly correlated plasma where the equilibration is driven by binary collisions [7]. However, a calculation analogous to that described here should also make it possible to extend our understanding of the equilibration process of a magnetized plasma into the strongly correlated regime.

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