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Stability analysis of a two-dimensional vortex pattern

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A normal mode analysis of perturbations on a stationary pattern formed by two-dimensional point vortices is carried out. The pattern is comprised of N equal vortices on a circle and one vortex of arbitrary strength in the center. The influence of an outer circular boundary is taken into account. The complete eigenfrequency spectrum is derived. [S1063-651X(97)11306-X]

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I. INTRODUCTION

The dynamics of two-dimensional point vortices has been the subject of theoretical and experimental studies for many years (see, e.g., Ref. [1] and literature cited therein). This idealized model has been used to describe two-dimensional (2D) vortex phenomena in various media including normal fluids [2], superfluid helium [3], and magnetized nonneutral plasmas [4]. Special emphasis has been placed on the stability of different vortex patterns, and a summary catalog covering many patterns has been compiled [5]. The history of this problem dates back to Thomson's [6] essay in 1883, where he carried out a normal mode analysis of perturbations on the ''classical'' vortex polygon. This system consists of N equal point vortices symmetrically spaced on the circumference of a circle. Thomson considered special cases $N=3$–7. Later Havelock [7] generalized the analysis to arbitrary N and also included the effect of a circular boundary. The same system, but with no boundary and with a point vortex of arbitrary strength located in the center of the circle, was partially treated in Ref. [8]. The most thorough analytical approach was developed in Ref. [9], where the whole set of normal modes and corresponding eigenfrequencies was derived for the case that includes the central vortex, but no boundary. However, the discussion of the bounded vortex pattern (which is the most relevant from the experimental point of view) was restricted to numerical results [9]. The present paper contains an analytical treatment of this case taking into account both central vortex and boundary effects. Besides their theoretical interest, the results of this work can be directly applied to recent experiments [10], in which the stability of the very same vortex pattern was studied in a magnetized electron plasma. Also stability issues considered in our paper substantially affect experiments [11] where the free relaxation of 2D turbulence in a magnetized electron column led to the formation of long-lived vortex crystals.

II. BASIC EQUATIONS

Consider a set of two-dimensional point vortices oriented parallel to the z direction and surrounded by an outer circular boundary of radius R. In a reference frame that rotates at frequency Ω around the z axis, the dynamics of such a system is governed by the Hamiltonian (see, e.g., Ref. [9])

$$H=\frac{1}{2}\sum_n \sum_{k\neq n} \gamma_n\gamma_k \ln|z_n-z_k|^2 -\frac{1}{2}\sum_n \sum_k \gamma_n\gamma_k \ln|z_n-\hat{z}_k|^2-\Omega L, \tag{1}$$

where $z_k=x_k+iy_k$ is a complex number representing the kth vortex position in the (x,y) plane, γ_k is the vortex strength (its total circulation), L is the angular momentum of the vortices

$$L=\sum_n \gamma_n|z_n|^2, \tag{2}$$

$\hat{z}_k=R^2/\overline{z}_k$ is the position of the image vortex due to the boundary, and the overbar denotes a complex conjugate. The first term in Eq. (1) describes the interaction energy between the vortices themselves, the second term results from the interaction between the vortices and the images, and the last term arises from the rotation. Hamilton's equations of motion are

$$i\gamma_n\dot{\overline{z}}_n=\frac{\partial H}{\partial z_n}, \tag{3}$$

having the explicit form

$$i\dot{\overline{z}}_n=-\Omega\,\overline{z}_n+\sum_{k\neq n}\frac{\gamma_k}{z_n-z_k}-\sum_k\frac{\gamma_k}{z_n-\hat{z}_k}. \tag{4}$$

Now consider the equilibrium ($\dot{z}_n^{(0)}=0$), in which N equal point vortices are symmetrically arranged in a circle,

$$z_n^{(0)}=\exp\{i\varphi_n\},\quad \varphi_n=2\pi n/N,\quad \gamma_n=1,\quad n=1,\ldots,N, \tag{5}$$

and one vortex of arbitrary strength is located in the center of the circle,

$$z_0^{(0)}=0,\quad \gamma_0=\gamma. \tag{6}$$

Here we use a normalization where each vortex in the circle has strength $\gamma_n=1$ and is positioned at radius $|z_n^{(0)}|=1$. Us-

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ing Eqs. (4)–(6), one can obtain the following expression for the equilibrium frequency Ω_0:

$$\Omega_0 = \gamma - \frac{N+1}{2} + \frac{N}{1-p^N}, \tag{7}$$

where $p = 1/R^2$. The derivation of Eq. (7) is completely analogous to that of Eq. (17) from Ref. [7].

To study the linear stability of perturbations on the polygon, we carry out a normal mode analysis and obtain the eigenfrequency spectrum. Introducing the infinitesimal displacements δz_k of the vortices from their equilibrium positions and linearizing Hamilton's equations in these perturbations yield

$$i\delta\dot{\overline{z}}_0 = -\Omega_0 \delta\overline{z}_0 - \sum_{k=1}^{N} \frac{\delta z_0 - \delta z_k}{z_k^{(0)2}} + \sum_{k=1}^{N} \frac{\delta z_0 - \delta\hat{z}_k}{\hat{z}_k^{(0)2}} + \gamma p \delta\overline{z}_0, \tag{8}$$

$$i\delta\dot{\overline{z}}_n = -\Omega_0 \delta\overline{z}_n - \sum_{\substack{k=1\\k\neq n}}^{N} \frac{\delta z_n - \delta z_k}{(z_n^{(0)} - z_k^{(0)})^2} + \sum_{k=1}^{N} \frac{\delta z_n - \delta\hat{z}_k}{(z_n^{(0)} - \hat{z}_k^{(0)})^2} - \gamma \frac{\delta z_n - \delta z_0}{z_n^{(0)2}} + \gamma p \; \delta\overline{z}_0. \tag{9}$$

Since the equilibrium polygon is symmetrical with respect to rotations, the normal modes can be searched for in the form of a superposition of azimuthal waves:

$$\delta z_0 = \alpha_c(t), \tag{10}$$

$$\delta z_n \exp\{-i\varphi_n\} = \sum_{m=0}^{N-1} \alpha_m(t) \exp\{im\varphi_n\}, \tag{11}$$

where φ_n is defined by Eq. (5). On the left-hand side of Eq. (11) we have introduced an additional exponential factor that produces a rotation of the reference frame through the angle $-\varphi_n$. After such a rotation, the equilibrium position of the nth vortex is on the x axis of the rotated frame. Thus each vortex is treated on an equal footing and the analysis is simplified.

It is worth noting that perturbations (10) and (11) leave H and L unchanged to first order. This may be significant for some excitation mechanisms since H and L are conserved under the dynamics. To understand that L does not suffer a first-order change (i.e., $\delta L = 0$), note that perturbation (11) does not change the mean-square radius of the vortex ring and that the displacement of the central vortex contributes to the angular momentum only in second order. Also, for this geometry, Aref [12] has shown that the perturbation in H is proportional to the perturbation in L [i.e., $\delta H = -(\Omega_0/2)\,\delta L = 0$]. Finally, we note that a perturbation that moves each vortex on the ring radially by the same amount, that is, makes a first-order change in the mean-square radius, simply establishes a new equilibrium and is uninteresting dynamically.

Inserting Eqs. (10) and (11) into Eqs. (8) and (9) and selecting terms with the same exponential factor yield equations for the α_m's. These equations can be divided into two groups. The first group describes the evolution of the α_m's for $m \neq 1, N-1$. These equations,

$$\dot{\overline{\alpha_m}} = i\overline{\alpha_m}\left[\Omega_0 - F_m - \frac{p}{(1-p)^2}\right] + i\alpha_{N-m}\left[G_m - T - \frac{p^2}{(1-p)^2} + \gamma\right], \tag{12}$$

show that $\overline{\alpha_m}$ is coupled only to α_{N-m}. Here we should note that α_0 is identical to α_N. The second group of equations describes the evolution of α_1, α_{N-1}, and α_c:

$$\dot{\overline{\alpha_c}} = i(\Omega_0 - \gamma p)\overline{\alpha_c} - iN\alpha_1 - ipN\overline{\alpha_{N-1}}, \tag{13}$$

$$\dot{\overline{\alpha_{N-1}}} = i\overline{\alpha_{N-1}}\left[\Omega_0 - F_{N-1} - \frac{p}{(1-p)^2}\right] + i\alpha_1\left[\gamma - T - \frac{p^2}{(1-p)^2}\right] - i\gamma p\overline{\alpha_c}, \tag{14}$$

$$\dot{\overline{\alpha_1}} = i\overline{\alpha_1}\left[\Omega_0 - F_1 - \frac{p}{(1-p)^2}\right] + i\alpha_{N-1}\left[\gamma - T - \frac{p^2}{(1-p)^2}\right] - i\gamma\alpha_c. \tag{15}$$

Here the following definitions have been used:

$$G_m = \sum_{k=1}^{N-1} \frac{1 - \exp\{i(m+1)\varphi_k\}}{(1 - \exp\{i\varphi_k\})^2}, \tag{16}$$

$$F_m = \sum_{k=1}^{N-1} \frac{p\exp\{i(1-m)\varphi_k\}}{(p - \exp\{i\varphi_k\})^2}, \tag{17}$$

$$T = \sum_{k=1}^{N-1} \frac{p^2}{(p - \exp\{i\varphi_k\})^2}. \tag{18}$$

The calculation of these sums is carried out in the Appendix. Following the terminology of Ref. [9], the modes governed by Eq. (12) will be referred to as "rational" modes and those governed by Eqs. (13)–(15) will be referred to as "cubic" modes.

It should be noted that for $N=2$ (vortices on a line) α_1 and α_{N-1} are identical and Eqs. (13)–(15) become invalid. This case requires special treatment and will be considered separately in Sec. IV.

III. RATIONAL MODES, $N>2$

Fixing some m and assuming $\alpha_m \sim \exp\{-i\omega t\}$, one obtains from Eq. (12), after routine algebra,

$$\omega_m^{(\pm)} = S_m \pm (P_m Q_m)^{1/2}, \quad m = 0, 2, \ldots, N-2, \tag{19}$$

where

$$S_m = -\frac{mN(p^{N-m} + p^m)}{2(1-p^N)} + \frac{N^2 p^{N-m}(1-p^{2m})}{2(1-p^N)^2}, \tag{20}$$

$$P_m=\frac{m(N-m)}{2}-\frac{N^2p^{N-m}(1-p^m)^2}{2(1-p^N)^2}+\frac{mN(p^{N-m}-p^m)}{2(1-p^N)}, \tag{21}$$

$$Q_m=2\gamma+N-1-\frac{m(N-m)}{2}+\frac{2Np^N}{1-p^N}+\frac{mN(p^{N-m}-p^m)}{2(1-p^N)} -\frac{N^2p^{N-m}(1+p^m)^2}{2(1-p^N)^2}. \tag{22}$$

The symmetry of coefficients (20)–(22),

$$S_m=-S_{N-m}, \quad P_m=P_{N-m}, \quad Q_m=Q_{N-m}, \tag{23}$$

leads to the relations

$$\omega_m^{(\pm)2}=\omega_{N-m}^{(\mp)2}. \tag{24}$$

Hence Eq. (19) determines $N-2$ values for the set of ω_m^2. The mode with $m=0$ has zero eigenfrequency and corresponds to the rigid rotation of the vortex pattern through a fixed angle. In the limit $\gamma=0$, eigenfrequencies (19) coincide with those of Eq. (26) from Ref. [7], while for the case with no boundary $p=0$ Eq. (19) reduces to Eq. (25) from Ref. [8].

As followed from Eq. (19), the mode is stable if $P_mQ_m>0$. Using the symmetry relations (23), we can show that $P_m>0$, so the stability criterion is simply

$$\min[Q_m]>0, \tag{25}$$

where

$$\min[Q_m]=\begin{cases} Q_{N/2} & \text{for even } N \\ Q_{(N\pm1)/2} & \text{for odd } N. \end{cases} \tag{26}$$

The explicit form of Eq. (25) is given by

$$\gamma>\frac{N^2-8N+8}{16}-\frac{Np^N}{1-p^N}+\frac{N^2p^{N/2}}{4(1-p^{N/2})^2} \tag{27}$$

for even N and by

$$\gamma>\frac{N^2-8N+7}{16}-\frac{Np^N}{1-p^N}-\frac{N(N-1)(p^{(N+1)/2}-p^{(N-1)/2})}{8(1-p^N)} +\frac{N^2p^{(N+1)/2}(1+p^{(N-1)/2})^2}{4(1-p^N)^2} \tag{28}$$

for odd N. If there is no central vortex $\gamma=0$, Eq. (27) follows from Eq. (29) of Ref. [7]. In the limit with no boundary $p=0$, criteria (27) and (28) are identical to Eqs. (29) from Ref. [8].

IV. CUBIC MODES, $N>2$

For solutions of the form $\exp\{-i\omega t\}$ [Eqs. (13)–(15)] yield

$$\left[S_1+\frac{1}{2}(P_1+Q_1)-\omega\right]\overline{\alpha_1}+\frac{1}{2}(Q_1-P_1)\alpha_{N-1}-\gamma\alpha_c=0, \tag{29}$$

$$\frac{1}{2}(P_1-Q_1)\overline{\alpha_1}+\left[S_1-\frac{1}{2}(P_1+Q_1)-\omega\right]\alpha_{N-1}+\gamma p\alpha_c=0, \tag{30}$$

$$N\overline{\alpha_1}+pN\alpha_{N-1}+(\gamma p-\Omega_0-\omega)\alpha_c=0, \tag{31}$$

with S_1,P_1,Q_1 given by Eqs. (20), (21), and (22), respectively. The corresponding eigenfrequency equation takes the form

$$\omega^3+c_2\omega^2+c_1\omega+c_0=0, \tag{32}$$

where

$$c_2=(\Omega_0-\gamma p-2S_1), \tag{33}$$

$$c_1=\gamma N(1-p^2)-P_1Q_1+S_1[S_1-2(\Omega_0-\gamma p)], \tag{34}$$

$$c_0=\frac{1}{2}\gamma N(1+p^2)(P_1+Q_1)+\gamma pN(P_1-Q_1) -\gamma N(1-p^2)S_1+(\gamma p-\Omega_0)(Q_1P_1-S_1^2). \tag{35}$$

With no central vortex $(\gamma=0)$, Eqs. (29) and (30) decouple from Eq. (31) and the eigenvalues are found to be

$$\omega_\pm=S_1\pm(P_1Q_1)^{1/2}, \tag{36}$$

$$\omega_*=-\Omega_0. \tag{37}$$

Frequencies (36), when combined with frequencies (19), lead to spectrum (26) from Ref. [7]. The formal solution ω_* is not of interest since for this normal mode $\alpha_1=\alpha_{N-1}=0$ and hence the vortex positions are not perturbed.

In the absence of the boundary $p=0$, coefficients (33)–(35) are substantially simplified, and one obtains from Eq. (32)

$$\omega_*=-\Omega_0, \quad \omega_\pm^2=\frac{1}{4}(N-1)^2-\gamma. \tag{38}$$

Here the frequency ω_* corresponds to the mode of the rigid displacement of the vortex pattern (in the laboratory frame of reference this mode has zero frequency). The eigenvalues $\omega_\pm$ follow from Eq. (35) of Ref. [9].

The exact analytical expressions for solutions of the cubic equation (19) are too cumbersome to be presented here. In the general case, the stability criterion for the cubic modes can be written as

$$4[3c_1-(c_2)^2]^3+[2(c_2)^3-9c_1c_2+27c_0]^2<0, \tag{39}$$

where c_2,c_1, and c_0 are defined in Eqs. (33), (34), and (35), respectively.

V. VORTICES ON LINE, $N=2$

As already noted in Sec. II, Eqs. (13)–(15) are not valid for the case $N=2$ since instead of two different amplitudes α_1 and α_{N-1} there is only α_1. Three amplitudes, namely, α_0,α_1, and α_c, appear in the problem. Equations for their time evolution are derived in the same way as for Eqs. (12)–(15). The amplitude α_0 is still governed by Eq. (12) and yields the eigenfrequency $\omega_0=0$, which corresponds to the rigid rotation of the vortex pattern through a fixed angle. The coupled equations for α_1 and α_c are found to be

$$\dot{\alpha}_1=i\bar{\alpha}_1\frac{1}{2}(Q_1+P_1)+i\alpha_1\frac{1}{2}(Q_1-P_1)-i\gamma p\bar{\alpha}_c-i\gamma\alpha_c, \tag{40}$$

$$\dot{\alpha}_c=i\bar{\alpha}_c\frac{1}{2}(\tilde{Q}+\tilde{P})+i\alpha_c\frac{1}{2}(\tilde{Q}-\tilde{P})-i2\alpha_1-i2p\bar{\alpha}_1, \tag{41}$$

with Q_1 and P_1 given by Eqs. (22) and (21), respectively, and

$$\tilde{Q}=\Omega_0-\gamma p+2(1-p^2),\quad \tilde{P}=\Omega_0-\gamma p-2(1-p^2). \tag{42}$$

The associated eigenfrequency equation yields

$$\omega^4+b_1\omega^2+b_0=0, \tag{43}$$

where

$$b_1=4\gamma(1-p^2)-Q_1P_1-\tilde{Q}\tilde{P}, \tag{44}$$

$$b_0=4\gamma^2(1-p^2)^2-2\gamma\tilde{Q}Q_1(1-p)^2-2\gamma\tilde{P}P_1(1+p)^2+Q_1P_1\tilde{Q}\tilde{P}. \tag{45}$$

Without the central vortex, $\gamma=0$, Eq. (40) decouples from Eq. (41) and the eigenfrequencies are given by

$$(\omega_1)^2=Q_1P_1,\quad (\omega_*)^2=\tilde{Q}\tilde{P}. \tag{46}$$

The first of these expressions is the same as Eq. (26) of Ref. [7]. The second one corresponds to a normal mode with no vortex displacements $\alpha_1=0$ and hence can be ignored in this limit.

The coefficients in Eqs. (40) and (41) are particularly simple for the case $p=0$ (no boundary) and one finds the eigenvalues

$$(\omega_*)^2=\Omega_0^2,\quad (\omega_1)^2=\Omega_0^2-(2+\gamma)^2. \tag{47}$$

Here ω_* corresponds to a mode of rigid displacement of the vortex pattern. The second expression is identical to Eq. (22) obtained in Ref. [8].

The calculation of the eigenfrequencies from Eq. (43) in the general case is elementary, but the final formulas are fairly cumbersome. It can be shown that $(b_1)^2-4b_0>0$ for arbitrary γ and $0<p<1$, so the stability criterion following from Eq. (43) reduces to the requirement

$$\pm[(b_1)^2-4b_0]^{1/2}>b_1. \tag{48}$$

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APPENDIX

The expression for G_m was calculated in Ref. [8] and is given by

$$G_m=\frac{1}{2}(m-1)[m-(N-1)]. \tag{A1}$$

Sums T and F_m can be written in the form

$$T=p^2\frac{\partial}{\partial p}\sum_{k=1}^{N-1}\frac{\exp\{-i\varphi_k\}}{1-p\exp\{-i\varphi_k\}}, \tag{A2}$$

$$F_m=p\frac{\partial}{\partial p}\sum_{k=1}^{N-1}\frac{\exp\{-im\varphi_k\}}{1-p\exp\{-i\varphi_k\}}. \tag{A3}$$

One can rewrite the sums entering Eqs. (A2) and (A3) as

$$-\frac{N-1}{p}+\frac{1}{p}\sum_{k=1}^{N-1}\frac{1}{1-p\exp\{-i\varphi_k\}}, \tag{A4}$$

$$\frac{1}{p^m}\sum_{k=1}^{N-1}\frac{1}{1-p\exp\{-i\varphi_k\}}-\frac{1}{p^m}\sum_{s=0}^{m}p^s\sum_{k=1}^{N-1}\exp\{-is\varphi_k\}, \tag{A5}$$

where the double sum in Eq. (A5) can be easily calculated. Equations (16) of Ref. [7] give

$$\sum_{k=1}^{N-1}\frac{1}{1-p\exp\{-i\varphi_k\}}=\frac{N}{(1-p^N)}-\frac{1}{(1-p)}. \tag{A6}$$

Substituting Eqs. (A4)–(A6) into Eqs. (A2) and (A3) yields, after some algebra,

$$T=-\frac{p^2}{(1-p)^2}+\frac{Np^N}{(1-p^N)^2}(p^N+N-1), \tag{A7}$$

$$F_m=-\frac{p}{(1-p)^2}+\frac{Np^{N-m}}{(1-p^N)^2}(mp^N+N-m). \tag{A8}$$

Equations (A7) and (A8) hold for $N>2$. For the case $N=2$, the corresponding expressions are trivially obtained from Eqs. (20)–(22):

$$G_0=\frac{1}{2},\quad G_1=0,\quad T=\frac{p^2}{(1+p)^2}, \tag{A9}$$

$$F_0=-\frac{p}{(1+p)^2},\quad F_1=\frac{p}{(1+p)^2}. \tag{A10}$$

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