Two-dimensional bounce-averaged collisional particle transport in a single species non-neutral plasma

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In this paper we describe a new theory of like particle collisional transport for a non-neutral plasma confined in a Penning trap. The theory is valid in the regime $\omega_b > \omega_E$, $\omega_b > \nu_c$, and $r_c < \lambda_D$ where ω_b is the axial bounce frequency, ω_E is the $\mathbf{E}\times\mathbf{B}$ rotation frequency, ν_c is the collision frequency, r_c is the cyclotron radius, and λ_D is the Debye length. In this regime each particle can be bounce averaged into a long rod and the transport understood as arising from the $\mathbf{E}\times\mathbf{B}$ drift motion of the rods due to long-range mutual interactions. This is a very different mechanism than is considered in the classical theory of transport, where a particle guiding center undergoes a step of order r_c as a result of a velocity scattering collision. For the parameter range considered, the new theory predicts transport rates that are orders of magnitude larger than those predicted by classical theory and that scale with magnetic field strength like $1/B$ rather than $1/B^4$. The new theory differs from a previous analysis of transport due to $\mathbf{E}\times\mathbf{B}$ drift interactions of charged rods, in that the finite length of the rods is taken into account. This enables transport to occur even for the case of an $\mathbf{E}\times\mathbf{B}$ drift rotation frequency that is a monotonic decreasing function of radius (as was the case in recent experiments).
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I. INTRODUCTION

Plasmas with a single sign of charge (e.g., pure ion plasmas and pure electron plasmas) are routinely confined for long times in Penning traps.[1,2] These simple traps are cylindrically symmetrical configurations in which the charges are confined radially by an axial magnetic field (typically uniform) and confined axially by electrostatic fields.[3] Over time, collisional processes drive the plasma to a state of global thermal equilibrium,[1,2,4] which is characterized by a temperature T and rotation frequency $\omega_R = v_\theta / r$ that are uniform over the whole plasma. Here, $v_\theta \hat{\theta}$ is the local fluid velocity and (r,θ,z) is a cylindrical coordinate system with its axis centered on the axis of the trap. Rotation is necessary for radial force balance; the inward magnetic force $ev_\theta B/c$ balances the outward forces due to space charge and pressure.

When the characteristic cyclotron radius and Debye length are both small compared to the radius of the plasma, and the plasma length is on the order of or less than the collisional mean-free path, the evolution to thermal equilibrium takes place on two well-separated time scales.[1] On the collisional time scale, thermal equilibrium is established along each field line. In fact, because of the cylindrical symmetry, the local thermal equilibria extend throughout thin cylindrical shells (say r to $r+\Delta r$). At this point in the evolution, the temperature and rotation frequency are functions of r. On a much longer time scale (the transport time scale), the different cylindrical shells come into thermal equilibrium with each other. Viscous forces acting on the shear in the rotational flow drive the system to a state of uniform ω_R, and heat conduction drives the system to a state of uniform T. One can show that uniform ω_R and T plus axial and radial force balance guarantee global thermal equilibrium.[1]

This paper contains a calculation of the coefficient of viscosity, which is the important coefficient in determining the particle transport. A recent paper[5] provided a calculation of the coefficient of heat conduction, and the predicted value is in good agreement with experiment.[6] The time scale for heat conduction is much shorter than that for particle transport, so when evaluating the particle transport, we treat the temperature as uniform.

As a first step it is useful to formalize the above description in a fluid dynamic framework for the particle transport.[1] For the frequency ordering $\omega_R \ll \Omega_c$, where Ω_c is the cyclotron frequency, the centrifugal force term is negligible, and radial force balance reduces to the form

$$0 = \frac{nev_\theta B}{c} - ne\frac{\partial \phi}{\partial r} - \frac{\partial}{\partial r} nT, \tag{1}$$

which then determines the local rotation frequency,

$$\omega_R = \frac{v_\theta}{r} = \frac{c}{Br}\frac{\partial \phi}{\partial r} + \frac{c}{Bner}\frac{\partial (nT)}{\partial r}. \tag{2}$$

The fluid velocity is the sum of an $\mathbf{E}\times\mathbf{B}$ drift and a diamagnetic drift. Here, n is the density and $p = nT$ is the pressure.

A shear in the rotational flow gives rise to the stress,

$$p_{r\theta} = -\eta r \frac{\partial \omega_R}{\partial r}, \tag{3}$$

where η is the coefficient of viscosity. Azimuthal force balance,

$$0 = -\frac{nev_r B}{c} - \frac{1}{r^2}\frac{\partial}{\partial r} r^2 p_{r\theta}, \tag{4}$$

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then yields the particle flux

$$\Gamma_r = n v_r = \frac{c}{eB}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \eta r \frac{\partial \omega_R}{\partial r}. \tag{5}$$

One can see that the flux scales with magnetic field strength like $1/B^2$ times the scaling of η with B.

In the typical operating regime for trapped non-neutral plasmas ($r_c \ll \lambda_D$), the classical theory of collisional transport[7] is not relevant.[8] The classical theory focuses on small impact parameter collisions (i.e., $\rho \lesssim r_c$), which scatter a particle velocity vector and produce a step in the particle guiding center position of about r_c. The predicted coefficient of viscosity[7] is $\eta \simeq n m \nu_c r_c^2$, where ν_c is the collision frequency, which implies a particle flux that scales like $1/B^4$. Experiments show that the predicted flux is orders of magnitude too small and has the wrong scaling with B. The problem is that classical theory effectively neglects the many collisions characterized by impact parameter in the range $r_c < \rho < \lambda_D$.[8] These large impact parameter collisions do not produce velocity scattering and, yet, dominate the transport.

Since the large impact parameter collisions can be treated in the guiding center drift approximation, we refer to theories that focus on these collisions as guiding center drift theories of transport. Different expressions for the viscosity are predicted, depending on the relative ordering of the frequencies ω_b, ω_R, and ν_c, where ω_b is the characteristic axial bounce frequency of the particles.

An initial analysis[8] assumed that each collision is uncorrelated with previous collisions, which makes sense when $\omega_b < \max(\omega_R, \nu_c)$. Two charges undergo **E×B** drift steps as they stream by one another, and the steps in different collisions are uncorrelated. For a Debye shielded interaction potential, collisions having an impact parameter in the range $\rho \sim \lambda_D$ dominate, and the predicted viscosity is of order $\eta \simeq n m \nu_c \lambda_D^2$, which exceeds the classical prediction by the large value $(\lambda_D/r_c)^2 \gg 1$. Since η is independent of B, the predicted particle flux scales like $1/B^2$.

However, experimentally it has turned out to be easier to observe transport to thermal equilibrium in plasmas with $\omega_b > \max(\omega_R, \nu_c)$, so the above theory does not apply. In two sets of experiments with pure electron plasmas,[1,9] the observed flux was large, but scaled like $1/B$ rather than $1/B^2$. For large axial bounce frequency, two particles collide many times producing a sequence of correlated **E×B** drift steps. This suggests a model in which each particle is bounce averaged into a long rod, and the rods then undergo two-dimensional (2-D) **E×B** drift transport due to their mutual interactions.

There has been much previous theoretical work on 2-D **E×B** drift theories of collisional transport. For example, it has been shown that any such model automatically yields transport rates that scale like $1/B$, in agreement with observations.[10,11] However, the earliest theories considered transport in a homogeneous (neutral) plasma for which the **E×B** rotation frequency $\omega_E(r)$ was spatially uniform (zero), whereas in the experiments with non-neutral plasmas $\omega_E(r)$ was large and nonuniform.

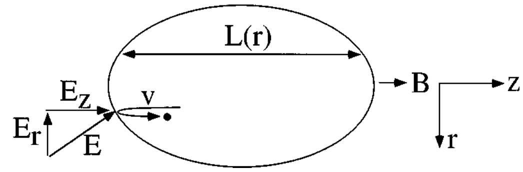

FIG. 1. Schematic view of particle dynamics in finite length plasma. A guiding center (dot) with speed v bounces off the end of a plasma of length $L(r)$.

The first analysis[12] that explicitly took into account the large and nonuniform $\omega_E(r)$, concluded that significant transport would occur only if $\omega_E(r)$ was not monotonic. The reasoning starts from the observation that only resonant interactions between the rods cause significant transport, implying that they must have the same rotation frequency: $\omega_E(r_1) = \omega_E(r_2)$ for rods at radii r_1 and r_2. If $\omega_E(r)$ is monotonic in r, this implies $r_1 = r_2$. However, conservation of angular momentum implies that the rods take equal and opposite radial steps when they **E×B** drift in their mutual Coulomb field; and if they start at the same radius these steps create no net particle flux.

For some years this analysis seemed to be consistent with experiment, since the first experiments[1] that observed $1/B$ scaling of the flux were carried out on plasmas with a nonmonotonic $\omega_E(r)$. However, a new set of experiments[9] were carried out on a plasma with a monotonic $\omega_E(r)$ and also found $1/B$ scaling. These new experiments have motivated a reexamination of the 2-D **E×B** drift theory, which is the subject of the present work.

In the previous theory,[12] the plasma was assumed for simplicity to have a length that was independent of radius. We now believe that for the new experiments this approximation is an oversimplification. If we allow the plasma to have finite length $L(r)$ that varies with radius, a new effect emerges: particle rotation frequencies now become a function of their axial speed v.[13] Consider the situation depicted in Fig. 1. A particle moves in a plasma that is equilibrated along the magnetic field and has a Debye length λ_D small compared to L, so that $\omega_E(r)$ is a function only of radius well within the plasma (i.e., more than a few Debye lengths from the plasma ends). However, as the particle approaches the end of the plasma, it feels a confining end potential that is a function of z and reverses its z velocity. Unless the plasma end is flat, this end potential depends on r as well as z, and the radial electric-field component of the end potential causes the particle to **E×B** drift in the θ direction, affecting the rotation frequency.

The larger the axial particle velocity, the larger the axial (and hence radial) impulse imparted by the end potential to the particle, and the larger the change in the rotation frequency. Thus, a particle's bounce-averaged rotation frequency can be written as $\omega(r,v) = \omega_E(r) + \Delta\omega(r,v)$ where $\Delta\omega(r,v)$ is the rotation frequency shift caused by bouncing off of the plasma end potentials.

In this more realistic picture of the particle motion, it is now possible for particles at different radii to have the same bounce-averaged rotation frequency, even when $\omega_E(r)$ is monotonic, provided that the particles have different axial speeds:

$$\omega_E(r_1)+\Delta\omega(r_1,v_1)=\omega_E(r_2)+\Delta\omega(r_2,v_2). \tag{6}$$

Since v_1 and v_2 differ, r_1 and r_2 can also differ, and the resonant interaction of the particles now causes transport to thermal equilibrium.

An order-of-magnitude estimate for the transport rate can be obtained from a straightforward argument. First, one can obtain $\Delta\omega(r,v)$ from the fact that the axial impulse imparted by the z component of the end electric field, E_z, must equal $2mv$. We picture the plasma as in Fig. 1; the Debye length λ_D is here assumed small compared to $L(r)$ so we can treat the particles as specularly reflecting off of the plasma ends. The geometry of the end potential implies that the radial component of the end electric field, E_r, is $E_z\partial(L/2)/\partial r$, so the radial impulse to the particle is $mv\,\partial L/\partial r$, causing the particle to $\mathbf{E}\times\mathbf{B}$ drift through an angle $\Delta\theta=-(v/r\Omega_c)\partial L/\partial r$. The frequency of collisions with each end is v/L, so averaged over many collisions with the ends a frequency shift to the rotation frequency develops, given by $\Delta\omega=\Delta\theta v/L$, or

$$\Delta\omega(r,v)=-\frac{v^2}{r\Omega_c L}\frac{\partial L}{\partial r}. \tag{7}$$

Let us now estimate the average radial distance between resonantly interacting particles. Taking $v_1^2-v_2^2\approx\overline{v}^2$, where $\overline{v}$ is the thermal speed $\sqrt{T/m}$, and taking $r_2=r_1+D$ in Eq. (6), we find that a Taylor expansion of $\omega_E(r_2)$ yields

$$D=\frac{\overline{v}^2}{r\Omega_c}\left|\frac{\partial\ \ln L/\partial r}{\partial\omega_E/\partial r}\right|. \tag{8}$$

We can employ this estimate to determine the coefficient of viscosity η in the plasma, which in turn yields the transport rate. General considerations imply that the viscosity scales as

$$\eta=mn\nu D^2, \tag{9}$$

where n is the density (in cm^{-3}), D is the average distance between interacting particles, which in this instance is given by Eq. (8), and v is a collision rate (the rate of transfer of momentum). The collision rate can be estimated as the usual classical collision frequency for two particle collisions, $\nu_c=n\overline{v}\overline{b}^2$, multiplied by the average number of collisions experienced by particles as they bounce between the ends, which is of order $\overline{v}/|Lr\partial\omega_E/\partial r|$. This enhancement arises because particles take on the order of this number of $\mathbf{E}\times\mathbf{B}$ steps in the same direction before they become decorrelated. Thus, we obtain the following estimate for the viscosity due to collisions between interacting rods:

$$\eta=mn\nu_c\frac{\omega_b}{|r\partial\omega_E/\partial r|}D^2. \tag{10}$$

Since η scales like B, the flux predicted by Eq. (10) scales like $1/B$.

II. KLIMONTOVITCH APPROACH TO 2-D E×B TRANSPORT

We now turn to a rigorous evaluation of the particle flux caused by bounce-averaged $\mathbf{E}\times\mathbf{B}$ drift collisions. This derivation is similar to that appearing in Ref. 12 for infinite length rods, but now we allow the plasma to be finite length.

Before we begin, we remind the reader that there are several rotation frequencies that appear in the analysis, and since it is easy to confuse them it may be useful to list them here. First, there is the $\mathbf{E}\times\mathbf{B}$ rotation frequency $\omega_E(r,z)$ experienced by particles. (In this section we make no assumptions concerning the size of the Debye length compared to the plasma length L, so ω_E is a function of both r and z.) Second, there is the total bounce-averaged rotation frequency $\omega(r,I)=\overline{\omega_E(r,z)}$ (the overbar denotes a bounce average). Here we introduce the bounce action I, which is a constant of the motion along a collisionless particle trajectory to be defined presently. Third, there is the rotation frequency of a fluid element, $\omega_R(r,z)=\omega_E(r,z)+(nm\Omega_c r)^{-1}\partial(nT)/\partial r$; the second term is the plasma diamagnetic drift. It is this third frequency that we expect to be uniform in thermal equilibrium.

We first define the Klimontovitch density for a system of M guiding centers,

$$N(\mathbf{r},p,t)=\sum_{j=1}^{M}\delta[\mathbf{r}-\mathbf{r}_j(t)]\delta[p-p(t)], \tag{11}$$

where $\mathbf{r}$ is the particle position and p is the axial momentum. This density satisfies the guiding-center Klimontovitch equation,

$$\frac{\partial N}{\partial t}(\mathbf{r},p,t)+\left(\frac{p}{m}\hat{z}-\frac{c}{B}\nabla\Phi\times\hat{z}\right)\cdot\nabla N(\mathbf{r},p,t)-e\frac{\partial\Phi}{\partial z}\frac{\partial N}{\partial p}(\mathbf{r},p,t)=0, \tag{12}$$

where $\Phi(\mathbf{r},t)$ is the electrostatic potential determined self-consistently via Poisson's equation,

$$\nabla^2\Phi=-4\pi e\int dpN(\mathbf{r},p,t). \tag{13}$$

Following the standard procedure, we define an average of N over an ensemble of initial conditions, $f(r,z,p,t)=\langle N(\mathbf{r},p,t)\rangle$, and a fluctuation $\delta N(\mathbf{r},p,t)=N(\mathbf{r},p,t)-f(r,z,p,t)$.

An equation for the average distribution f can be obtained by averaging Eq. (12):

$$\frac{\partial f}{\partial t}+\left(\frac{p}{m}\hat{z}+\frac{c}{B}\frac{\partial\phi}{\partial r}\hat{\theta}\right)\cdot\nabla f-e\frac{\partial\phi}{\partial z}\frac{\partial f}{\partial p}=\nabla\cdot\left(\left\langle\frac{c}{B}\nabla\delta\phi\times\hat{z}\,\delta N\right\rangle\right)+\frac{\partial}{\partial p}\left\langle e\frac{\partial\delta\phi}{\partial z}\delta N\right\rangle, \tag{14}$$

where $\phi(r,z,t)$ is the self-consistent mean field plasma potential, given by

$$\nabla^2\phi=-4\pi e\int dpf, \tag{15}$$

and where $\delta\phi$ is the fluctuation in the potential, given by $\delta\phi=\Phi-\phi$, and determined by δN via the linearized Poisson equation,

$$\nabla^2\delta\phi=-4\pi e\int dp\,\delta N. \tag{16}$$

The first term on the right-hand side of Eq. (14) is the divergence of the dissipative particle flux and the second term the dissipative axial momentum flux. In this paper we focus on the radial particle flux,

$$\Gamma_r(r,z,p,t)=-\frac{c}{Br}\left\langle\frac{\partial\delta\phi}{\partial\theta}(r,\theta,z,t)\,\delta N(r,\theta,z,p,t)\right\rangle. \tag{17}$$

Note that we expect the azimuthal component of the dissipative particle flux to vanish, and the radial component to be independent of θ due to symmetry of the equilibrium in θ.

An equation for the evolution of the fluctuations is obtained by subtracting Eq. (14) from Eq. (12) and dropping quadratic terms in the fluctuations:

$$\frac{\partial\delta N}{\partial t}+\left(\frac{p}{m}\hat{z}+\frac{c}{B}\frac{\partial\phi}{\partial r}\hat{\theta}\right)\cdot\boldsymbol{\nabla}\delta N-e\frac{\partial\phi}{\partial z}\frac{\partial\delta N}{\partial p}$$
$$=\frac{c}{Br}\frac{\partial\delta\phi}{\partial\theta}\frac{\partial f}{\partial r}+e\frac{\partial\delta\phi}{\partial z}\frac{\partial f}{\partial p}. \tag{18}$$

In order to solve Eq. (18) it is useful to switch to action-angle variables $(z,p)\rightarrow(\psi,I)$, where I is the bounce action, running from 0–∞, and ψ is the canonically conjugate angle variable, running from $0-2\pi$. These coordinates are defined in terms of the mean-field potential $\phi(r,z)$:

$$I=\oint p\,dz/2\pi, \tag{19}$$

where the momentum p varies in z along the particle's trajectory according to

$$p=\sqrt{2m[H-e\phi(r,z)]}, \tag{20}$$

and where H is a constant of integration equal to the single-particle energy (mean-field Hamiltonian). The angle variable ψ then follows from Eq. (19) and the fact that the transformation is canonical.[14] In terms of the new coordinates, Eq. (18) becomes

$$\frac{\partial}{\partial t}\delta N+\omega_b(r,I)\frac{\partial\delta N}{\partial\psi}+\omega(r,I)\frac{\partial\delta N}{\partial\theta}$$
$$=\frac{c}{Br}\frac{\partial\delta\phi}{\partial\theta}\frac{\partial f(r,I)}{\partial r}+e\frac{\partial\delta\phi}{\partial\psi}\frac{\partial f(r,I)}{\partial I}, \tag{21}$$

where ω_b is the bounce frequency, and ω is the particle rotation frequency, defined by

$$\omega_b=\frac{\partial H(r,I)}{\partial I},\quad \omega=\frac{c}{eBr}\frac{\partial H(r,I)}{\partial r}, \tag{22}$$

and $H(r,I)$, the single-particle Hamiltonian in action-angle coordinates, is found by inverting Eq. (19) and solving for H.

In Eq. (21) we have assumed that f is a function only of the constants of the motion, r and I, so that f is a solution of the equilibrium Vlasov equation obtained by neglecting the dissipative fluxes (i.e., the rhs) in Eq. (14). In particular, we neglect any time dependence of f in Eq. (21). This is because Eq. (14) implies that f evolves in time on a slow transport time scale, and this slow time dependence can be neglected when evaluating the rapid time evolution of the fluctuations.

We are interested in 2-D transport in the limit where ω_b is large compared to ω, so it is useful to define a bounce-averaged density fluctuation and potential,

$$\overline{\delta N}=\int_0^{2\pi}d\psi\,\delta N,\quad \overline{\delta\phi}=\int_0^{2\pi}\frac{d\psi}{2\pi}\delta\phi. \tag{23}$$

Note that $\overline{\delta N}$ is defined without the factor of 2π so that $\int\overline{\delta N}dI$ is the fluctuation in the number of rods per unit area: $\int\overline{\delta N}dI=\int\delta N\,dp\,dz$.

In order to obtain self-consistent equations for the bounce-averaged fluctuations, it is necessary to solve Eq. (21) exactly before any bounce averaging is performed and then make an expansion of the exact solution in ω/ω_b. The solution to Eq. (21) for δN can be obtained by Laplace transformation in time and Fourier transformation in θ and ψ:

$$\delta N(\mathbf{r},p,t)=\sum_{l,n}\frac{e^{il\theta+in\psi}}{2\pi}\int\frac{ds}{2\pi i}\delta N(r,I,l,n,s)e^{st}, \tag{24a}$$

$$\delta\phi(\mathbf{r},t)=\sum_{l,n}e^{il\theta+in\psi}\int\frac{ds}{2\pi i}\delta\phi(r,I,l,n,s)e^{st}. \tag{24b}$$

When Eqs. (24) are compared to Eq. (23), we see that the Fourier–Laplace transform of the bounce-averaged fluctuations are related to the $n=0$ coefficients of Eqs. (24):

$$\overline{\delta N}(r,I,l,s)=\delta N(r,I,l,n=0,s) \tag{25a}$$

and

$$\overline{\delta\phi}(r,I,l,s)=\delta\phi(r,I,l,n=0,s). \tag{25b}$$

In terms of these Fourier–Laplace components, the solution of Eq. (21) is

$$\delta N(r,I,l,n,s)=\frac{\dfrac{ilc}{Br}\dfrac{\partial\overline{f}}{\partial r}+ine\dfrac{\partial\overline{f}}{\partial I}}{s+il\omega(r,I)+in\omega_b(r,I)}\delta\phi(r,I,l,n,s)$$
$$+\sum_{j=1}^{M}\frac{\delta(r-r_j)\delta(I-I_j)e^{-il\theta_j-in\psi_j}}{2\pi r_j[s+il\omega(r,I)+in\omega_b(r,I)]}, \tag{26}$$

where the sum over j is an explicit expression for the initial fluctuation and where $\overline{f}=2\pi f$ is defined so that $\int\overline{f}dI=\int f\,dz\,dp$, the mean number of rods per unit area.

Up to now we have not made any approximations involving bounce averaging. We now introduce the bounce-averaging approximation to Eq. (26). Since $\omega\ll\omega_b$, and the time scale of the bounce-averaged fluctuations is of order ω^{-1}, we can approximate the $n\neq0$ terms in δN as

$$\delta N(r,I,l,n,s)=e\,\frac{\partial\bar{f}/\partial I}{\omega_b(r,I)}\,\delta\phi(r,I,l,n,s),\quad n\neq 0. \tag{27a}$$

We make no other approximations, and, in particular, the $n=0$ term in Eq. (26) is treated exactly:

$$\delta N(r,I,l,n=0,s)=\frac{(ilc/Br)\partial\bar{f}/\partial r}{s+il\omega(r,I)}\,\delta\phi(r,I,l,n=0,s)+\sum_{j=1}^{M}\frac{\delta(r-r_j)\delta(I-I_j)e^{-il\theta_j}}{2\pi r_j[s+il\omega(r,I)]}. \tag{27b}$$

In Eq. (27a) we have neglected $n\neq 0$ terms in the initial conditions, since they are of $O(\omega/\omega_b)$; we have neglected $[ilc/Br]\partial\bar{f}/\partial r$ in favor of in $e\partial\bar{f}/\partial I$ for the same reason; and we have also dropped the $s+il\omega$ term from the denominator. Equation (27a) is a linearized Boltzmann response of the $n\neq 0$ density fluctuations to the potential fluctuations. This can be seen by noting that Eq. (22) implies $(\partial\bar{f}/\partial I)/\omega_b=\partial\bar{f}/\partial H$, and for a Boltzmann distribution $\partial\bar{f}/\partial H=-\bar{f}/T$. Thus, for a Boltzmann distribution Eq. (27a) becomes $\delta N=-e\,\delta\phi\bar{f}/T$. The Boltzmann form applies to $n\neq 0$ components of δN because the bounce-averaged fluctuations evolve sufficiently slowly so that charges have time to equilibrate along the magnetic field lines.

In order to obtain an equation for the potential fluctuations, we substitute Eq. (27) into the linearized Poisson's equation, Eq. (16). The solution for $\delta\phi$ can then be written in terms of a Green's function, $\chi(r,z,r_j,z_j,l,s)$, that satisfies

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{l^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)\chi(r,z,r_j,z_j,l,s)+4\pi e^2\int\frac{dp}{2\pi}\frac{\partial\bar{f}}{\partial H}\left(\chi(r,z,r_j,z_j,l,s)-\int\frac{d\bar{\psi}}{2\pi}\chi(r,\bar{z},r_j,z_j,l,s)\right)+\frac{4\pi ecil}{Br}\int\frac{dp}{2\pi}\frac{\partial\bar{f}(r,I)/\partial r}{s+il\omega(r,I)}\times\int\frac{d\bar{\psi}}{2\pi}\chi(r,\bar{z},r_j,z_j,l,s)=\frac{\delta(r-r_j)\delta(z-z_j)}{r}, \tag{28}$$

where here $\bar{z}=\bar{z}(I,\bar{\psi},r)$ and $I=I(z,p,r)$.

We will presently observe that we have need only of the bounce-averaged part of $\delta\phi$, which involves only the bounce average of the Green's function:

$$\delta\bar{\phi}(r,I,l,s)=-\sum_j\frac{4\pi e}{2\pi}\frac{e^{-il\theta_j}}{s+il\omega(r_j,I_j)}\,\bar{\chi}(r,I,r_j,I_j,l,s), \tag{29a}$$

where the bounce average of χ is carried out over both z and z_j:

$$\bar{\chi}(r,I,r_j,I_j,l,s)=\int\frac{d\psi d\psi_j}{(2\pi)^2}\,\chi(r,z,r_j,z_j,l,s), \tag{29b}$$

and where $z=z(I,\psi,r)$ and $z_j=z_j(I_j,\psi_j,r_j)$. The bounce-averaged density fluctuation $\delta\bar{N}$ then follows from Eq. (27b) and Eqs. (25).

We now employ these expressions for the fluctuations in order to determine the cross-field particle flux. In particular, we will consider the bounce average of the particle flux for particles with a particular value of the action I:

$$\bar{\Gamma}_r(r,I)\equiv\int d\psi\Gamma_r[r,z(I,\psi,r),p(I,\psi,r)]=\frac{c}{Br}\sum_{l,n}\int\frac{ds\,d\bar{s}\,e^{(s+\bar{s})t}}{(2\pi i)^2}\langle\delta N(r,I,l,n,s)\times(il)\,\delta\phi(r,I,-l,-n,\bar{s})\rangle, \tag{30}$$

where we have used Eqs. (17) and (24) and have taken into account the symmetry in θ of the flux. However, the $\omega/\omega_b\ll 1$ approximations that led to Eq. (27a) also imply that $n\neq 0$ terms in Eq. (30) do not contribute to the flux. This is because Eq. (27a) implies that the $n\neq 0$ terms in Eq. (30) are proportional to $\Sigma_{l,n,n\neq 0}l\langle\delta\phi(r,I,l,n,t)\delta\phi(r,I,-n,-l,t)\rangle$. This sum is antisymmetric under $l\to -l$, $n\to -n$, and therefore vanishes. Thus, only the $n=0$ (bounce-averaged) fluctuations contribute to $\bar{\Gamma}_r$.

Substituting for these bounce-averaged fluctuations using Eqs. (27b) and (29a), and averaging over the uncorrelated initial conditions yields

$$\bar{\Gamma}_r(r,I)=-\frac{c}{Br}\left(\frac{4\pi e}{2\pi}\right)^2\sum_l\int 2\pi r_j dr_j dI_j\bar{f}(r_j,I_j)\times\int\frac{ds\,d\bar{s}}{(2\pi i)^2}e^{(s+\bar{s})t}\times\frac{(il)\bar{\chi}(r,I,r_j,I_j,-l,\bar{s})}{[s+il\omega(r_j,I_j)][\bar{s}-il\omega(r_j,I_j)]}\times\left(\frac{\delta(r-r_j)\delta(I-I_j)}{4\pi er_j}-\frac{cil}{Br}\frac{\partial\bar{f}(r,I)}{\partial r}\frac{\bar{\chi}(r,I,r_j,I_j,l,s)}{s+il\omega(r,I)}\right). \tag{31}$$

We now make the Bogoliubov ansatz, assuming that fluctuations decay to their asymptotic form on a time scale fast compared to the collision rate. This involves an evaluation of the s and $\bar{s}$ integrals using the Cauchy residue method retaining only the poles along the imaginary axis, since any poles in the Green's functions are assumed to be heavily damped. The result is

$$\overline{\Gamma}_r = -\frac{c}{Br}\left(\frac{4\pi e}{2\pi}\right)^2 \sum_l \int 2\pi r_j dr_j dI_j \overline{f}(r_j,I_j)\left(-\frac{l\delta(r-r_j)\delta(I-I_j)}{4\pi e r_j}\,\mathrm{Im}\,\overline{\chi}^*\,[r,I,r_j,I_j,l,-il\omega(r_j,I_j)]\right.$$
$$\left.+\frac{cl^2}{Br}\frac{\partial\overline{f}(r,I)}{\partial r}|\overline{\chi}[r,I,r_j,I_j,l,-il\omega(r_j,I_j)]|^2\pi\delta\{l[\omega(r,I)-\omega(r_j,I_j)]\}\right). \tag{32}$$

In order to obtain a symmetric form for the flux, we rewrite the first term in the large brackets with the aid of Eq. (28). If we set $s=-il\omega(r_j,I_j)+\varepsilon$ in Eq. (28) and multiply both sides by r Im $\chi^*[r,z,r_j,z_k,l,-il\omega(r_j,I_j)]$, the result after integrating over r and z, integrating by parts, applying the Plemelj formula to the resonant denominator, and bounce averaging in z_j and z_k, is

$$\mathrm{Im}\;\overline{\chi}^*[r_j,I_j,r_j,I_j-il\omega(r_j,I_j)]$$
$$=\int \frac{d\psi_j d\psi_k}{(2\pi)^2}\,\mathrm{Im}\int rdrdz\left\{\chi^*[r,z,r_j,z_k,l,-il\omega(r_j,I_j)]\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{l^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)\chi[r,z,r_j,z_j,l,-il\omega(r_j,I_j)]\right.$$
$$+4\pi e^2\int\frac{dp}{2\pi}\chi^*[r,z,r_j,z_k,l,-il\omega(r_j,I_j)]\frac{\partial\overline{f}}{\partial H}\left(\chi[r,z,r_j,z_j,l,-il\omega(r_j,I_j)]-\int\frac{d\overline{\psi}}{2\pi}\chi[r,\overline{z},r_j,z_j,l,-il\omega(r_j,I_j)]\right)$$
$$\left.+\frac{4\pi ecl}{Br}\int\frac{dp}{2\pi}\chi^*[r,z,r_j,z_k,l-il\omega(r_j,I_j)]\frac{\partial\overline{f}}{\partial r}\int\frac{d\overline{\psi}}{2\pi}\chi[r,\overline{z},r_j,z_j,l,-il\omega(r_j,z_j)]i\pi\delta(l[\omega(r_j,I_j)-\omega(r,I)])\right\}, \tag{33}$$

where $z_j=z(I_j,\psi_j,r_j)$, $z_k=z(I_j,\psi_k,r_j)$, $\overline{z}=z(I,\overline{\psi},r)$.

The imaginary part of the first term in the braces is zero because the integrals over ψ_j and ψ_k bounce average the Green's functions in their second z argument. Integrations by parts in r and z then prove that the product of the Green's functions is real. Similarly, the imaginary part of the second term is also zero. For the third term we note that $dzdp=dId\psi$, and the ψ integration bounce averages χ^* over z, so we arrive at

$$\mathrm{Im}\;\overline{\chi}^*[r_j,I_j,r_jI_j,-il\omega(r_j,I_j)]$$
$$=\frac{4\pi ecl}{B}\int drdI\,\frac{\partial\overline{f}(r,I)}{\partial r}|\overline{\chi}[r,I,r_j,I_j,l,$$
$$-il\omega(r_j,I)]|^2\pi\delta\{l[\omega(r,I)-\omega(r_j,I_j)]\}, \tag{34}$$

which can then be substituted into Eq. (32) to yield

$$\overline{\Gamma}_r(r,I)=\frac{1}{2r}\left(\frac{4\pi ec}{B}\right)^2\sum_l |l|\int_0^\infty dI_j$$
$$\times\int_0^\infty r_j dr_j|\overline{\chi}[r,I,r_j,I_j,l,-il\omega(r_j,I_j)]|^2$$
$$\times\delta[\omega(r_j,I_j)-\omega(r,I)]$$
$$\times\left(\frac{\overline{f}(r,I)}{r_j}\frac{\partial\overline{f}(r_j,I_j)}{\partial r_j}-\frac{\overline{f}(r_j,I_j)}{r}\frac{\partial\overline{f}(r,I)}{\partial r}\right). \tag{35}$$

Equation (35), the bounce-averaged radial flux of rods caused by bounce-averaged 2D **E×B** collisions in a finite length plasma, is the main result of the paper.

The flux conserves total canonical angular momentum $P_\theta=\int 2\pi rdrdI\overline{f}(r,I,t)m\Omega_c r^2/2$, which follows from the fact that the integral $\int rdrdIr\overline{\Gamma}_r(r,I)$ vanishes due to the antisymmetry of the integrand under interchange of (r,I) and (r_j,I_j). The flux also conserves the total electrostatic and kinetic energy of the plasma,

$$E=\int 2\pi rdrdpdzf(r,z,p,t)\left(\frac{p^2}{2m}+\frac{1}{2}e\phi_p(r,z,t)\right.$$
$$\left.+e\phi_e(r,z)\right), \tag{36}$$

where we have broken $\phi(r,z,t)$ into an intrinsic plasma potential ϕ_p and an external potential ϕ_e: $\phi=\phi_p+\phi_e$. The external potential is responsible for the finite length of the plasma and arises from voltages applied to surrounding cylindrically symmetric electrodes. The intrinsic plasma potential is the solution of Eq. (15) for given f assuming that the surrounding electrodes are all grounded. The time dependencies arise from the slow evolution of f on a transport time scale. Energy conservation can be derived by taking the time derivative of Eq. (36), using Eqs. (14), (15), and (22) and neglecting the axial momentum flux $\langle e(\partial\delta\phi/\partial z)\delta N\rangle$ in Eq. (14). Then using the relation $dpdz=dId\psi$ we obtain, after integrating by parts,

$$\frac{dE}{dt}=-m\Omega_c\int 2\pi rdrdIr\overline{\Gamma}_r(r,I)\omega(r,I). \tag{37}$$

This integral vanishes because Eq. (35) implies that it is antisymmetric under interchange of (r,I) and (r_j,I_j). Evidently neglect of the axial momentum flux does not affect energy conservation, implying that this flux produces a rearrangement of the parallel action among the collection of rods without exchange of energy with the other degrees of freedom. Finally, it is not difficult to show that the flux increases an entropy function S,

$$S=-\int 2\pi r dr dI \overline{f}(r,I,t) \ln \overline{f}(r,I,t). \tag{38}$$

One can also see that the flux vanishes when the plasma is in a state of global thermal equilibrium. This state is characterized by uniform temperature T and fluid rotation frequency ω_R. To show this, assume that $\overline{f}(r,I)$ is Maxwellian, but with arbitrary radial dependence given by

$$\overline{f}(r,I)=Ce^{-[H(r,I)-m\Omega_c\int^r r\omega_R(r)dr]/T}, \tag{39}$$

where C is a constant. This form for the distribution is chosen because it is Maxwellian in velocity, but has a fluid rotation $\omega_R(r)$ with arbitrary radial dependence. This can be seen by transforming from the variables (ψ,I) back to variables (z,p) using Eq. (20):

$$f(r,z,p)=\frac{\overline{f}(r,I)}{2\pi} = \frac{C}{2\pi}\, e^{-[p^2/2m+e\phi(r,z)-m\Omega_c\int_0^r r\omega_R(r)dr]/T}. \tag{40}$$

Equation (40) displays the Maxwellian form of the distribution. The Maxwellian form is required because small-impact parameter velocity-scattering collisions, not explicitly accounted for in the theory, drive the distribution toward this form.

The connection between the fluid velocity and the function $\omega_R(r)$ that appears in Eq. (40) can be obtained by differentiating Eq. (40) with respect to r after integrating over momentum. One finds that

$$\omega_R(r)=\frac{c}{Br}\frac{\partial\phi}{\partial r}+\frac{T}{m\Omega_c r}\frac{\partial \ln n}{\partial r}, \tag{41}$$

where $n(r,z)=\int dp f(r,z,p)$ is the particle density. Equation (41) agrees with Eq. (2) when T is constant. Note that the rotation frequency is a function only of r, although both the $\mathbf{E}\times\mathbf{B}$ and diamagnetic drifts separately depend on both r and z. This is a consequence of the Boltzmann equilibrium along the field lines assumed in Eq. (40).

Substituting Eq. (39) in Eq. (35), the expression in the square brackets becomes

$$\frac{\overline{f}(r,I)}{r_j}\frac{\partial}{\partial r_j}\overline{f}(r_j,I_j)-\frac{\overline{f}(r_j,I_j)}{r}\frac{\partial}{\partial r}\overline{f}(r,I) = \frac{m\Omega_c}{T}[\omega_R(r_j)-\omega_R(r)-\omega(r_j,I_j) + \omega(r,I)]\overline{f}(r,I)\overline{f}(r_j,I_j), \tag{42}$$

where we have employed Eq. (22). However, the resonance condition $\omega(r,I)=\omega(r_j,I_j)$ implies that the last two terms in the square brackets cancel, so Eq. (35) becomes

$$\overline{\Gamma}_r(r,I)=\frac{1}{2r}\left(\frac{4\pi ec}{B}\right)^2\sum_l |l|\int_0^\infty dI_j r_j dr_j|\overline{\chi}[r,I,r_j,I_j,l, -il\omega(r_j,I_j)]|^2\delta[\omega(r_j,I_j)-\omega(r,I)] \times\frac{m\Omega_c}{T}[\omega_R(r_j)-\omega_R(r)]\overline{f}(r,I)\overline{f}(r_j,I_j). \tag{43}$$

Thus, the flux vanishes when the fluid rotation frequency $\omega_R(r)$ is uniform, that is, when the system is in a state of thermal equilibrium.

III. LOCAL COEFFICIENT OF VISCOSITY

It is useful to employ the following simple model when evaluating Eq. (43). Assuming that the Debye length is small compared to the plasma length $L(r)$, $\phi(r,z)=\phi(r)$ within the plasma and the axial motion consists of particles undergoing specular reflection off the plasma ends. This model was discussed in the Introduction in relation to Fig. 1. In this case the bounce action is $I=|p|L(r)/\pi$, the single-particle mean-field energy is

$$H(r,I)=\frac{\pi^2 I^2}{2mL^2(r)}+e\phi(r), \tag{44}$$

and the bounce-averaged distribution of rods is

$$\overline{f}(r,I)=\frac{2\pi n(r)}{\sqrt{2\pi mT}}\, e^{-\pi^2 I^2/2mTL^2(r)}, \tag{45}$$

where $n(r)$ is the particle density, and the factor of 2 in the numerator arises because I is non-negative. Note that $\int_0^\infty \overline{f} dI=n(r)L(r)$ is the number of rods per unit area, as expected. The particle rotation frequency, $\omega=c/eBr \times\partial H/\partial r$, is

$$\omega(r,I)=\omega_E(r)-\frac{\pi^2 I^2}{m^2\Omega_c r L^3(r)}\frac{\partial L}{\partial r}, \tag{46}$$

where $\omega_E(r)=(c/Br)\partial\phi/\partial r$ is the $\mathbf{E}\times\mathbf{B}$ rotation frequency, in agreement with the heuristic result derived in the Introduction, Eq. (7).

Also, note that the bounce-averaged Green's function, $\overline{\chi}$, is independent of I and I_j in this model. The bounce average over an orbit now merely involves an integral over z between the plasma's ends and is independent of the value of a particle's action.

Applying Eqs. (45) and (46) to the expression for the flux, Eq. (43), we find that it reduces to the expression derived in Ref. 12 for the case where the plasma has flat ends, $\partial L/\partial r=0$. In this case the flux vanishes when $\omega_E(r)$ is monotonic in r because the resonance condition implies $r=r_j$. However, when $\partial L/\partial r\neq 0$ the flux is nonzero because particles with $r\neq r_j$ can satisfy the resonance condition $\omega(r,I)=\omega(r_j,I_j)$. As discussed in the Introduction, the typical radial distance D between such particles is given by Eq. (8).

Note that when $\omega_E(r)$ and $L(r)$ vary on a scale of order the plasma radius R, Eq. (8) implies that the interaction distance D is small, of order λ_D^2/R, so interacting particles are closely spaced in radius. This allows us to make a local approximation for the flux and obtain a local coefficient of viscosity, using the following math identity: consider the radial integral of the antisymmetric function $\omega_R(r_j)-\omega_R(r)$ multiplied by a symmetric function $S[(r+r_j)/2,|r-r_j|]$ that is sharply peaked in the second argument:

$$J=\int dr_j S\left(\frac{r+r_j}{2},|r-r_j|\right)[\omega_R(r_j)-\omega_R(r)]. \tag{47}$$

Taylor expansion with respect to slowly varying arguments then implies that to lowest order

$$J=\frac{\partial}{\partial r}\left(\int_{-\infty}^{\infty}d\Delta r S(r,|\Delta r|)\,\frac{\Delta r^2}{2}\,\frac{\partial\omega_R}{\partial r}\right). \tag{48}$$

Note that our expression for the flux, Eq. (43), has this form when integrated over I:

$$\int_0^\infty dI\overline{\Gamma}_r(r,I)=\frac{1}{2r^2}\left(\frac{4\pi ec}{B}\right)^2\frac{m\Omega_c}{T}\int dr_j S\left(\frac{r+r_j}{2},\right.$$
$$\left.|r-r_j|\right)[\omega_R(r_j)-\omega_R(r)], \tag{49}$$

where

$$S(R,|\Delta r|)=\sum_l |l|\int_0^\infty dIdI_j r r_j|\overline{\chi}[r,r_j,l,$$
$$-il\omega(r_j,I_j)]|^2$$
$$\times\delta[\omega(r_j,I_j)-\omega(r,I)]\overline{f}(r,I)\overline{f}(r_j,I_j),$$

$$r=R-\Delta r/2,\qquad r_j=R+\Delta r/2. \tag{50}$$

When Eq. (48) is applied to Eqs. (49)–(50) we find that the flux of rods is given by the local expression

$$\int_0^\infty dI\overline{\Gamma}_r(r,I)=\frac{1}{2r^2}\left(\frac{4\pi ec}{B}\right)^2\frac{m\Omega_c}{T}\frac{\partial}{\partial r}\sum_l |l|\int_{-\infty}^{\infty}d\Delta r\frac{\Delta r^2}{2}$$
$$\times\int_0^\infty dIdI_j r_+r_-|\overline{\chi}[r_-,r_+,l,$$
$$-il\omega(r_+,I_j)]|^2\delta[\omega(r_+,I_j)$$
$$-\omega(r_-,I)]\overline{f}(r_-,I)\overline{f}(r_+,I_j)\,\frac{\partial\omega_R}{\partial r}, \tag{51}$$

where $r_+=r+\Delta r/2$ and $r_-=r-\Delta r/2$. We now perform the integral over the action I_j, substituting for the distribution functions using Eq. (45) and the particle rotation frequency using Eq. (46) to obtain

$$\int_0^\infty dI\overline{\Gamma}_r(r,I)=\frac{1}{2r^2}\left(\frac{4\pi ec}{B}\right)^2\frac{m\Omega_c}{T}\frac{\partial}{\partial r}\sum_l |l|\int_{-\infty}^{\infty}d\Delta r\frac{\Delta r^2}{2}\,r_+r_-$$
$$\times\int dI\,\frac{2\pi n_+n_-}{mT}\,e^{(-\pi^2/2mT)[(I^2/L_-^2)(1+r_+L_+L_-'/r_-L_-L_+')-(m^2\Omega_c/\pi^2)(r_+L_+/L_+')\Delta\omega_E]}$$
$$\times|\overline{\chi}[r_-,r_+,l,-il\omega(r_-,I)]|^2\,\frac{1}{\left|\dfrac{2\pi^2L_+'}{m^2\Omega_c r_+L_+^2}\right|\sqrt{\dfrac{r_+L_+L_-'}{r_-L_-L_+'}\,\dfrac{I^2}{L_-^2}-\dfrac{m^2\Omega_c}{\pi^2}\,\dfrac{r_+L_+}{L_+'}\,\Delta\omega_E}}\,\frac{\partial\omega_R}{\partial r}, \tag{52}$$

where $\Delta\omega_E=\omega_E(r_+)-\omega_E(r_-)$, $n_{(-)}^{+}=n(r_{(-)}^{+})$, $L_{(-)}^{+}=L(r_{(-)}^{+})$, primes denote differentiation with respect to r, and the integral over I runs only over the region of $I>0$ where the argument of the square root is positive. The square root is an explicit expression of I_j/L_+, obtained by solving $\omega(r_+,I_j)=\omega(r_-,I)$ using Eq. (46).

Since Δr is assumed throughout to be small, and $n(r)$, $\omega(r)$, and $L(r)$ are slowing varying on the scale of Δr, we keep only the lowest-order terms in Δr to obtain $\Delta\omega_E=\Delta r\partial\omega_E/\partial r$, $r_+\simeq r_-\simeq r$. Thus, Eq. (52) becomes

$$\int_0^\infty dI\overline{\Gamma}_r(r,I)$$
$$=\frac{1}{2r^2}\left(\frac{4\pi ec}{B}\right)^2\frac{m^2\Omega_c^2}{T^2}\frac{\partial}{\partial r}\sum_l |l|\,\frac{r^3n^2(r)L^3(r)}{\pi|L'(r)|}$$
$$\times\int_0^\infty dv e^{-v^2/\overline{v}^2}\int_{-\infty}^{Dv^2/\overline{v}^2}d\Delta r\,\frac{\Delta r^2}{2}\,\frac{e^{\Delta r/2D}}{\sqrt{v^2-\overline{v}^2\Delta r/D}}$$
$$\times|\overline{\chi}[r-\Delta r/2,r+\Delta r/2,l,-il\omega(r,I)]|^2\frac{\partial\omega_R}{\partial r}, \tag{53}$$

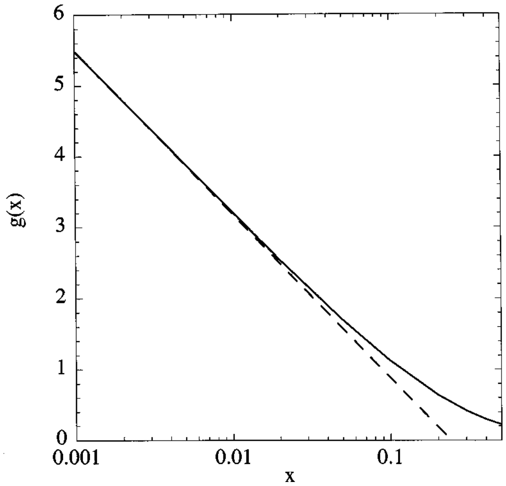

FIG. 2. The function $g(x)$. The solid line is a numerical evaluation of Eq. (58); the dashed line is the small x asymptotic form given by Eq. (60).

where D is the interaction distance defined in Eq. (8), and where we have changed integration variables from I to velocity $v=\pi I/mL$ and have interchanged the Δr and v integrals. Note that the substitution $r_+=r_-=r$ is not made in $\overline{\chi}$ since, for large $|l|$, $\overline{\chi}(r-\Delta r/2,r+\Delta r/2,l,-il\omega)$ is not slowly varying in Δr.

In order to make further progress we need an explicit form for the bounce-averaged Green's function $\overline{\chi}$. In general, this requires a numerical solution to Eq. (28). However, for a long thin plasma column, for which $D\ll R\ll L$ we will see that the sum over l is dominated by large l values, for which the interaction is short range and almost unshielded. The Fourier transformed Green's function between unshielded rods separated by a small distance is

$$\overline{\chi}(r,r',l)=-\frac{1}{2|l|L_>}\left(\frac{r_<}{r_>}\right)^l, \tag{54}$$

where $r_{\binom{>}{<}}$ is the larger (lesser) of r and r' and $L_>$ is the larger of $L(r)$ and $L(r')$. Provided that the distance Δr between the rods is small compared to their average radial position, the Green's function simplifies to

$$\chi(r-\Delta r/2,r+\Delta r/2,l)=-\frac{e^{-l|\Delta r|/r}}{2|l|L(r)}. \tag{55}$$

When this form for $\overline{\chi}$ is used in Eq. (53), the integrals over Δr and v can be performed analytically, yielding an equation for the flux of rods involving the local coefficient of viscosity $\overline{\eta}$:

$$\int_0^\infty dI\overline{\Gamma}_r=\frac{c}{eBr^2}\frac{\partial}{\partial r}r^3\overline{\eta}\frac{\partial\omega_R}{\partial r}, \tag{56}$$

where

$$\overline{\eta}=\frac{8\pi^2e^4}{T}\frac{D^2n^2(r)}{|r\omega_E'|}g\left(\frac{2D}{r}\right), \tag{57}$$

and where the function $g(2D/r)$ is

$$g\left(\frac{2D}{r}\right)=\sum_{l=1}^{\infty}\frac{h(2Dl/r)}{l}, \tag{58}$$

TABLE I. Table of values of $g(x)$.

x	g
0.001	5.48
0.002	4.80
0.005	3.89
0.01	3.21
0.02	2.54
0.05	1.70
0.1	1.12
0.2	0.638
0.3	0.418
0.4	0.295
0.5	0.218

where the function h is defined as

$$h(x)=\frac{1}{4\pi}\int_0^\infty dv\,e^{-v^2}\int_{-v^2}^{\infty}ds\,\frac{s^2e^{-s/2-x|s|}}{\sqrt{v^2+s}}$$
$$=\frac{2}{\pi}\,\frac{(1+8x^2)\ln\left(\frac{\sqrt{2x+1}+\sqrt{2x-1}}{\sqrt{2x+1}-\sqrt{2x-1}}\right)-6x\sqrt{4x^2-1}}{(4x^2-1)^{5/2}}. \tag{59}$$

An asymptotic analysis in the limit of small D/r yields

$$\lim_{D/r\to 0}g\left(\frac{2D}{r}\right)=\ln\left(\frac{r}{2D}\right)-2+\gamma, \tag{60}$$

where $\gamma=0.577...$ is Euler's constant. A numerical evaluation of $g(x)$ is shown in Fig. 2 and values are tabulated in Table I. The origin of the logarithmic divergence in D/r is easy to understand. When D/r is small, we can approximate

$$\chi(r,r'l)\sim\chi(r,r,l)=-\frac{1}{2L|l|}, \tag{61}$$

where the last step follows from Eq. (54).

Substituting for $|\chi|^2$ into Eq. (53) then yields the log divergence in the sum over l, cut off at $l_{\max}=r/D$, beyond which the approximation in Eq. (61) breaks down. As we stated previously, the sum is dominated by large l values to logarithmic accuracy, consistent with the approximation of an unshielded Green's function in Eq. (54).

Comparing the flux of rods, Eq. (56) [in units of 1/(m s)], to the particle flux, Eq. (5) [in units of 1/(m$^2-$s)], we see that the bounce-averaged viscosity $\overline{\eta}$ has different units than the usual fluid viscosity η. If L were constant as a function of radius so that it could be moved through the derivatives in Eq. (56), we would find that $\eta=\overline{\eta}/L$. Since L is not constant, Eq. (56) is the proper form for the flux of rods, in that this form satisfies the conservation of momentum, particle number, and energy.

However, in order to connect Eq. (57) to our previous estimate, Eq. (10), it is useful for scaling purposes to take $\eta=\overline{\eta}/L$. In this case Eq. (57) implies

$$\eta=8\pi^2mn\frac{\omega_b}{|r\omega_E'|}\nu_cD^2g(2D/r), \tag{62}$$

in agreement with the scaling estimate, Eq. (10).

IV. DISCUSSION

Equations (56) and (57) determine the radial flux of rods (bounce-averaged electrons) due to long-range guiding center collisions in a non-neutral plasma whose length varies with radius. This local form of the viscous particle transport rests on the assumption that the interaction distance D, defined in Eq. (8), is small compared to the scale length of the density and rotation frequency gradients. When these gradients are on the scale of the plasma radius R, this assumption is equivalent to the assumption that the Debye length is small: Eq. (8) implies $D/R \sim (\lambda_D/R)^2$, so we require $\lambda_D/R \ll 1$.

In deriving Eq. (57) we also assumed that the plasma was long: $L \gg \lambda_D \gg D$, so that the specular reflection approximation of Sec. III is valid, and so that the interaction between rods is given by Eq. (54), an unshielded interaction for the large azimuthal mode numbers that dominate the viscosity to logarithmic order. Finally, the analysis was also aided by the assumption that we consider the viscosity at a radial position $r \gg D$, so that the exponential form of the interaction, Eq. (55), is valid.

On the other hand, in order for our analysis to be relevant we also require that this 2-D viscosity be larger than the flux caused by uncorrelated collisions, discussed in Ref. 8. There the viscosity was shown to scale as $\eta \sim mnv_c\lambda_D^2$, and comparing this viscosity to the 2-D viscosity of Eq. (62) shows that the 2-D viscosity dominates only if

$$\frac{\omega_B}{|r\omega_E'|} > \left(\frac{\lambda_D}{D}\right)^2 \gg 1. \tag{63}$$

Thus, Eqs. (56)–(57) are relevant only when the plasma is well into the 2-D regime, where the bounce frequency is much larger than the $\mathbf{E}\times\mathbf{B}$ drift rotation frequency. However, the second inequality in Eq. (63) can be relaxed if we are willing to employ the full nonlocal kinetic equation, Eq. (43), because in Eq. (43) D need not be small compared to λ_D. Of course, solving for the flux in this case becomes considerably more difficult as a practical matter, since the fully shielded bounce-averaged Green's function that appears in Eq. (43) must be evaluated numerically. Nevertheless, such calculations are within the realm of possibility and may well be required in order to provide a detailed comparison of theory to present experiments for which $1 < \omega_B/\omega_E \lesssim 20$.[9]

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