Correction to Electron Plasma Mode Frequency Formula

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Abstract

A correction to the Prasad-O'Neil formula for non-axisymmetric Trivelpiece-Gould plasma wave frequencies is presented. The proper analytic limit for pure electron plasmas presented here differs from the limit analyzed previously, and the frequencies differ by the ratio of Bessel function zeros $j_{m_\theta}/j_{m_{\theta-1}}$.

The simple Prasad-O'Neil formula[1] for Trivelpiece-Gould plasma wave frequencies $\omega = \omega_{m_\theta,m_z,m_r}$ mentioned (but not used) in several experimental papers[2–5] is appropriate for pure ion plasmas, but not for pure electron plasmas. This note provides a corrected expression for z-periodic, non-axisymmetric waves appropriate to cylindrical pure electron plasmas in strong magnetic fields. For these electron plasmas, the mode frequencies ω are large compared to the plasma rotation ω_E, so the approximation $m_\theta\omega_E/\omega \ll 1$ is more appropriate than the previously analyzed approximation $\omega - m_\theta\omega_E \approx 0$. The electron approximation gives a revised frequency formula which substitutes Bessel function zero $j_{m_\theta-1,m_r}$ for j_{m_θ,m_r}.

The azimuthal, axial, and radial mode numbers (m_θ, m_z, m_r) were originally[1] denoted by (ℓ, k, n). Taking wave potential and density $\{\phi, \delta n\} \sim \exp(i\ell\theta + ikz - i\omega t)$, the cold fluid equations plus Poisson's equation give the mode equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} - \frac{\ell^2\phi}{r} - \varepsilon_{zz}k^2\phi + 2\ell\frac{\omega_E}{\omega_*}\frac{\phi}{rn_0}\frac{\partial n_0}{\partial r} = 0\ .$$

As in Ref. 1, we define $\varepsilon_{\rm zz} \equiv 1 - \omega_p^2/\omega_*^2 \equiv -\alpha^2$ for $r < R_p$, and $\varepsilon_{\rm zz} = 0$ for $r > R_p$. The plasma rotation is $\omega_E \equiv 2\pi\, ec\, n_0/B$, and the plasma-frame wave frequency is $\omega_* \equiv \omega - \ell\,\omega_E$. The (un-normalized) eigenfunction is given by

$$\phi = \begin{cases} J_\ell(\alpha\, kr) & \text{for } r < R_p \\ I_\ell(kr)\, K_\ell(k\, R_w) - K_\ell(kr)\, I_\ell(k\, R_w) & \text{for } r > R_p\ . \end{cases}$$

We consider $\ell > 0$, and make approximations $k\, R_p \ll 1$, and $R_p \ll R_w$. Matching the eigenfunction at $r = R_p$ gives

$$\alpha\, k\, R_p \frac{J_\ell'(\alpha\, k\, R_p)}{J_\ell(a\, kR_p)} = \ell\frac{(R_p/R_w)^{2\ell} + 1}{(R_p/R_w)^{2\ell} - 1} - 2\ell\frac{\omega_E}{\omega_*},$$

which is Eq. (11) of Ref. 1.

For pure ion plasmas, the limit of $\omega_* \approx 0$ is appropriate, making the $2\ell\omega_E/\omega_*$ term dominant, and thereby requiring $J_\ell(\alpha\, k\, R_p) \approx 0$. This gives $\alpha\, kR_p = j_{\ell n}$, where $j_{\ell n}$ is the n$^{\rm th}$ root of $J_\ell(x) = 0$. The frequencies are then

$$\omega - \ell\,\omega_E = \pm\omega_p\frac{kR_p}{j_{\ell n}}\ ,$$

as presented in Ref. 1 and (inappropriately) reproduced in later references treating electron plasmas.[2–5]

For typical pure electron plasmas in strong magnetic fields, the correct limit is $\omega_* \gg \ell\,\omega_E$, making $2\ell\,\omega_E/\omega_* \approx 0$. Using $xJ'_\ell(x) = -\ell J_{\ell-1}(x) + xJ_{\ell-1}(x)$, we obtain $J_{\ell-1}(\alpha\,k\,R_p) \approx 0$, giving

$$\omega - \ell\,\omega_E = \pm\omega_p \frac{kR_p}{j_{\ell-1,n}} \; .$$

For $\ell = 1$ and $n = 1$, this increases the predicted frequency, by as much as $j_{10}/j_{00} = 3.832/2.405 = 1.59$ when ω_E is negligible.

For completeness, we note that the eigenfunction $\delta n_{\ell,n}$ for non-uniform $n_0(r)$ is

$$\delta n_{\ell,n}(r) \propto n_0(r)\, J_\ell(j_{\ell-1,n}\, r/R_p) \; .$$

In finite-length devices, with $k = m_z\,\pi/L_p$, one obtains

$$\omega - \ell\,\omega_E \propto N_L^{1/2} L_p^{-1} \; ,$$

where $N_L \equiv N/L_p \sim n_0\pi\,R_p^2$ is the line density. Thus, the mode frequencies in this limit are independent of the detailed density profile $n_0(r)$.

Experimentally, the corrected frequency formula agrees with measurements on $L_p \sim$ 50 cm electron columns in the CamV apparatus at $B > 1$ kG, to within the line density uncertainty of $\lesssim 5\%$. For shorter columns, end-effect corrections[6] or numerical eigenvalue solutions may be required.

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[1] S.A. Prasad and T.M. O'Neil, Phys. Fluids **26**, 665 (1983), Eq. (A4).

[2] E.M. Hollmann, "Experimental Studies of Cross-Magnetic-Field Transport in Nonneutral Plasmas," Ph.D. dissertation, Univ. of Calif., San Diego (1999), Eq. (B.5).

[3] E.M. Hollmann, F. Anderegg, and C.F. Driscoll, Phys. Plasmas **7**, 1767 (2000), Eq. (4).

[4] F. Anderegg, E.M. Hollmann, and C.F. Driscoll, Phys. Rev. Lett. **81**, 4875 (1998), Eq. (1).

[5] Y. Soga, Y. Kiwamoto, and N. Hashizumi, Phys. Plasmas **13**, 052105 (2006), Eq. (8).

[6] J.K. Jennings, R.L. Spencer, and K.C. Hansen, Phys. Plasmas **2**, 2630 (1995), length corrections for $m_\theta = 0$.