Electronic and Positronic Guiding-Center Drift Ions

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A novel type of guiding-center drift ion is described. These ions occur only in strong magnetic fields. They consist of a neutral atom to which either an electron or positron is weakly bound, at a sufficiently large radius that it may be described by $\mathbf{E} \times \mathbf{B}$ drift dynamics. Such ions may occur naturally in astrophysical plasmas and may have been formed in recent antihydrogen experiments, where their presence would provide proof that deeply bound $\bar{\mathrm{H}}$ atoms are being created.

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This Letter describes a novel type of ion that exists only in strong magnetic fields: a guiding-center ion. These ions consist of a neutral particle (an atom, molecule, or nanoparticle) to which a single electron or positron is bound at a distance large compared to the electron cyclotron radius. Stable guiding-center ions can be formed using any neutral particle, provided that the magnetic field is sufficiently large.

The outer electron or positron in a guiding-center ion executes $\mathbf{E} \times \mathbf{B}$ drift rotation around the neutral particle. The electric field arises from the attractive van der Waals interaction between the neutral particle and the outer charge, along with the dipole and quadrupole moments (if any) of the neutral particle. Without the magnetic field, these short-range interactions would lead to unstable classical orbits that either escape to infinity or spiral into the central neutral particle [1]. However, the applied magnetic field stabilizes the orbits.

It is conceivable that guiding-center ions may occur in strong magnetic fields associated with astrophysical phenomena such as neutron stars. A considerable effort over the years has gone into studying the state of neutral matter in such strong magnetic fields [2], and while there has been work on the properties of negative ions in such fields, little is known concerning positron attachment, as far as we know. The scaling of bound state negative ion energies has been considered [3], and there are several variational calculations of the ground state and the first few excited states of H^- and other negative ions [4]. There has also been work describing positive ions (atoms that are missing electrons, not atoms with positrons attached) and ionic molecules [2,5].

Here, we focus on the regime of guiding-center ions where the dynamics of the outer charge is quasiclassical–i.e., the ion is excited well above the ground state. In this regime these ions have aspects in common with Rydberg atoms [6], a major difference being that here the interaction potential for the outer electron is not Coulombic.

The dynamics of these ions is similar to that of guiding-center hydrogen atoms, where an electron $\mathbf{E} \times \mathbf{B}$ drifts in the Coulomb potential of a central proton [7]. Antimatter guiding-center atoms have been observed to form in recent experiments that create atomic antihydrogen by recombination of positrons and antiprotons [8,9]. We will show that it is possible that guiding-center ions are also formed in these experiments, although they have not yet been observed.

We first examine a simple classical version of a guiding-center ion, consisting of a classical guiding-center atom to which an electron or positron of charge $e_2 = \pm e$ is attached at a cylindrical radius r_2. The electron in the atom, charge $e_1 = -e$, circles the central proton (assumed fixed at the origin) at a radius $r_1 < r_2$; but the presence of e_2 affects the e_1 orbit. A uniform magnetic field $\mathbf{B}$ in the z direction is assumed to be sufficiently strong so that the electron cyclotron frequency Ω_{ce} is larger than all other frequencies in the problem, allowing the guiding-center approximation to be applied. [Later we will relax this assumption, showing that it need not hold for the inner electron. We also assume that r_1 is much greater than the Bohr radius; conditions on e_2 for the validity of this classical description are discussed in relation to Eq. (3).]

The Hamiltonian for this system is

$$H(\mathbf{r}_1, v_{z1}, \mathbf{r}_2, v_{z2}) = \frac{1}{2} m v_{z1}^2 + \frac{1}{2} m v_{z2}^2 - \phi(|\mathbf{r}_1|) \mp \phi(|\mathbf{r}_2|) \pm \phi(|\mathbf{r}_1 - \mathbf{r}_2|), \quad (1)$$

where $\mathbf{r}_1 = (r_1, \theta_1, z_1)$ and $\mathbf{r}_2 = (r_2, \theta_2, z_2)$, v_{z1}, and v_{z2} are the respective velocities parallel to the magnetic field, $\phi(r) = e^2/r$, and the upper and lower signs correspond to $e_2 = \mp e$. The equations of motion for e_1 and e_2 are $m\ddot{z}_i = -\partial H/\partial z_i$, $\dot{r}_i = -c/(e_i B r_i)\partial H/\partial \theta_i$, and $\dot{\theta}_i = c/(e_i B r_i)\partial H/\partial r_i$, $i = 1, 2$. This system has only two constants of the motion, the energy H and the quantity $r_1^2 \pm r_2^2$, and so is not generally integrable. (However, if motion is restricted to the xy plane, the dynamics is integrable, and will be considered in a future article.)

If e_1 is tightly bound at radius $r_1 \ll r_2$, the drift motion of e_1 is rapid, and we can make a two-time-scale approximation that considers the dynamics of e_1 assuming e_2 is stationary. This dynamics can then be used to find the motion of e_2, averaged over the rapid motion of e_1. For

example, one finds that to lowest nontrivial order, $\dot{\theta}_1 = -ce/B\bar{r}_1^3 \equiv -\omega_1$, and $r_1 = \bar{r}_1 \mp r_2\bar{r}_1^3\cos(\theta_1 - \theta_2)/(r_2^2 + z_2^2)^{3/2}$, where $\bar{r}_1$ is the initial orbit radius of e_1. It is assumed here that $\theta_1 = \theta_2$ at $t = 0$. If we further assume that the z position of e_1 responds adiabatically to the motion of e_2, neglecting the rapid axial bounce motion of e_1, the resulting slow-timescale motion of e_2 is Hamiltonian in form, with a Hamiltonian

$$\langle H\rangle(\mathbf{r}_2, v_{z2}) = \frac{1}{2}mv_{z2}^2 \mp eQ\frac{z_2^2 - r_2^2/2}{(r_2^2 + z_2^2)^{5/2}} - e^2\frac{\alpha_r r_2^2 + \alpha_z z_2^2}{(r_2^2 + z_2^2)^3} + O(\bar{r}_1^4), \qquad (2)$$

where $Q = e\bar{r}_1^2/2$ is the axial quadrupole moment of a ring of charge $-e$ with radius $\bar{r}_1$ (the time average of the e_1 orbit in the absence of e_2) and $(\alpha_r, \alpha_z) = (5/4, 1/2)\bar{r}_1^3$ are coefficients in the van der Waals interaction between the guiding-center atom and e_2. This attractive interaction arises from the induced dipole moment of the atom in the electric field of e_2 [10].

If e_2 moves only in the xy plane, the charge's binding energy (i.e., the value of $-\langle H\rangle$) is $E_b = \mp eQ/2r_2^3 + e^2\alpha_r/r_2^4$. The charge executes $\mathbf{E}\times\mathbf{B}$ drift rotation about the central atom with a rotation frequency $\omega_2 = (c/B)(3Q/2r_2^5 \mp 4\alpha_r e/r_2^6)$. If e_2 is a positron, the dynamics is stable in the z direction: a positron perturbed slightly from the xy plane will perform harmonic oscillations in z with frequency $\omega_{z2} = [[9eQ/2r_2^5 + (6\alpha_r - 2\alpha_z)e^2/r_2^6]/m]^{1/2}$.

However, if e_2 is an electron, the quadrupole term is repulsive and z motion is stable at $z = 0$ only if $r_2 \le 4e(3\alpha_r - \alpha_z)/9Q = 26\bar{r}_1/9$. For $r_2 > 26\bar{r}_1/9$, the outer electron is stable at an axial location determined by the potential minimum in $\langle H\rangle$ as a function of z_2 (the curve in Fig. 1). To the order given in $\bar{r}_1$, this minimum is at $z_2 = \pm(\sqrt{3/2}\,r_2 - 19\bar{r}_1/3\sqrt{15})$, with binding energy $E_b = 2\sqrt{2/5}\,e^2\bar{r}_1^2/25r_2^3 + 16e^2\bar{r}_1^3/125r_2^4$, rotation frequency $\omega_2 = (ec/B)(6\sqrt{2/5}\,\bar{r}_1^2/25r_2^5 + 64\bar{r}_1^3/125r_2^6)$, and harmonic axial frequency $\omega_{z2} = [(e^2/m)(36\sqrt{2/5}\,\bar{r}_1^2/125r_2^5 + 1696\bar{r}_1^3/3125r_2^6)]^{1/2}$.

We have tested these predictions by numerically simulating a guiding-center ion using the guiding-center Hamiltonian of Eq. (1). Results are shown in Figs. 1 and 2 for various values of the parameter $\omega_1/\Omega_{ce} = mc^2/\bar{r}_1^3B^2$. (This parameter must be less than unity in order for the guiding-center approximation to be valid for e_1.) By varying the initial r_2 value, we find stable guiding-center ions can exist with binding energy (Fig. 2), rotation frequency, and axial bounce frequency (not shown) that follow the previous predictions, provided that $r_2/\bar{r}_1$ is sufficiently large. (Binding energy is determined by slowly moving e_2 along a prescribed trajectory to its equilibrium position, starting from a large distance, and measuring the change in energy of the system.)

Furthermore, we find that, as expected, an outer positron is stable only in the xy plane, whereas an outer electron stably orbits the central particle at a height z_2 that follows the previous prediction (Fig. 1).

However, when $r_2/\bar{r}_1 \lesssim 10$, results begin to diverge noticeably from the theory predictions. There are initial values of $r_2 > 26\bar{r}_1/9$ where the outer electron is stable at $z_2 = 0$, at least over several hundred periods of the inner electron motion (Fig. 1). Furthermore, if $r_2/\bar{r}_1$ is too small initially, the guiding-center ion displays chaotic behavior. When e_2 is an electron, this typically results in its loss, with an increase in the binding of e_1 to compensate for the lost binding energy of e_2. When e_2 is a positron, the electron and positron often form a pair that $\mathbf{E}\times\mathbf{B}$ drifts in parallel away from the central charge (a "drifting pair" [10]).

This chaotic behavior occurs because the two-timescale approximation breaks down: the most rapid dynamics of e_2 (i.e., the z motion) has roughly the same frequency as the rotational motion of the inner charge, i.e., $\omega_{z2} \sim \omega_1$. Using our previous theory expressions for these frequencies, and neglecting constants of order unity, this implies that the ion is stable only if $r_2/\bar{r}_1 \gtrsim (\Omega_{ce}/\omega_1)^{1/5}$.

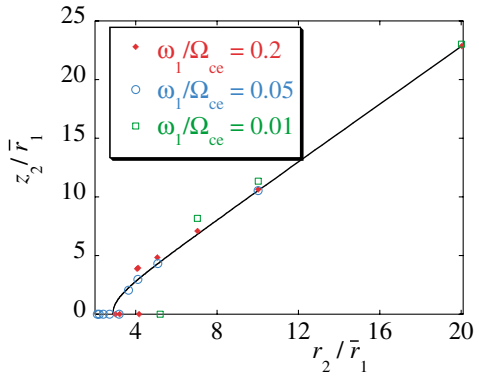

FIG. 1 (color online). Axial equilibrium position for the outer electron in electronic classical guiding center ions. Curve is theory, dots are simulation results.

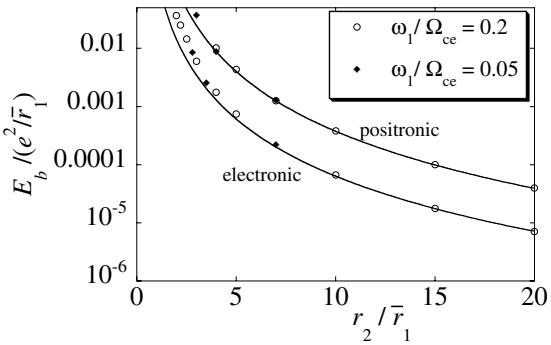

FIG. 2. Binding energy (affinity) for positronic (upper curve) and electronic (lower curve) classical guiding center ions. Dots are simulation results, curves are theory.

The boundary between stable and unstable ions was found numerically by performing many simulations with different randomly chosen initial conditions and values of ω_1/Ω_{ce}, starting with both z_1 and z_2 within $0.01\bar{r}_1$ of the xy plane and $|v_{z1}|$ and $|v_{z2}|$ less than $0.0005ec/B\bar{r}_1^2$. For any given initial value of $r_2/\bar{r}_1$ and ω_1/Ω_{ce}, the fraction f of times the outer charge was lost in a time $100/\omega_1$ or less was computed over many simulations. For $e_2 = -e$, contours of constant f are shown in Fig. 3. The $f = 0.8$ contour is fitted by $r_2/\bar{r}_1 = 2.0(\Omega_{ce}/\omega_1)^{0.21}$, in close agreement to our previous estimate.

We now relax the assumption that the central positive charge is fixed at the origin. The behavior of the system now depends on the ratios Ω_{ci}/ω_2 and $V_i/(r_2\,\omega_2)$, where $\Omega_{ci} = eB/Mc$ is the ion cyclotron frequency, and M and V_i are the mass and initial velocity of the central positive charge. When these ratios are small, the approximation that the central charge is stationary is a good one: e_2 executes roughly circular orbits about the central charge, and the entire system executes relatively slow circular ion cyclotron orbits with frequency Ω_{ci}. However, if $V_i/(r_2\,\omega_2) \gtrsim 1$, the positive charge and e_1 run away from e_2, leaving it behind. In what follows we assume that $V_i = 0$; i.e., the central positive charge is initially stationary.

If $\omega_2/\Omega_{ci} \lesssim 1$, the ion cyclotron motion of the system is distorted, and in particular r_2 oscillates with time; the magnitude of these oscillations tends to increase with decreasing ω_2/Ω_{ci}. If during these oscillations $r_2/\bar{r}_1$ decreases below the previously determined stability limit, e_2 is lost. Thus, stable guiding-center ions exist primarily in the range $\omega_2/\Omega_{ci} \gtrsim 1$. This rough inequality defines a maximum value of $r_2/\bar{r}_1$ for stability: $r_2/\bar{r}_1 \lesssim [C(M/m)\omega_1/\Omega_{ce}]^{1/5}$, where C is a constant of order unity. Therefore, the heavier the central charge, or the larger the value of ω_1/Ω_{ce}, the larger the range of stable $r_2/\bar{r}_1$ values.

This scaling was tested by running many simulations, allowing the central positive charge to move. Random initial conditions were integrated forward for times up to $10^5/\omega_1$. Initial values of r_2 and ω_1/Ω_{ce} resulting in stable H^- ions are shown on Fig. 3 as points. The clustering is consistent with our scaling results, taking $C = 5$ (the dashed line labeled H^-). Also shown are the scaling curves for $M = 4m_p$ and $131m_p$ (labeled He^- and Xe^-).

One can see from Fig. 3 that stable H^- ions exist primarily in the range $\omega_1/\Omega_{ce} \gtrsim 0.2$, a range for which guiding-center dynamics is a poor approximation for e_1. However, even if $\omega_1/\Omega_{ce} > 1$, e_2 can still be described by guiding-center dynamics, and stable guiding-center ions can still be found. An example of a stable guiding-center H^- ion for which $\omega_1/\Omega_{ce} = 10$ is shown in Fig. 4; now e_1's dynamics is treated exactly, and motion of the central proton is allowed. For this simulation, the binding energy of the outer electron was determined to be $0.017\,e^2/\bar{r}_1$, equal to 10 K when $B = 6T$.

Thus, the observation of H^- ions in a recombining hydrogen plasma (or the antimatter equivalent) provides an indicator of the binding depth of the hydrogen (or antihydrogen) atoms: only if these atoms are bound such that $\omega_1/\Omega_{ce} \gtrsim 0.2$ can H^- ions form. In a 6 Tesla magnetic field, this inequality corresponds to an atomic binding energy $e^2/r\¯_1$ deeper than 14 meV, roughly 5 times the binding energy estimated using E-field reionization diagnostics in current experiments [8,11].

For stability, the outer electron must be bound with energy $E_b \lesssim 0.04e^2/\bar{r}_1 = 6$ K when $e^2/\bar{r}_1 = 14$ meV (see Fig. 2); and E_b can be even greater for greater $e^2/\bar{r}_1$. Thus, guiding-center ions may form in current antihydrogen experiments for which the plasma temperature T_p satisfies $T_p \lesssim E_b$, provided that $e^2/\bar{r}_1 \gtrsim 14$ meV $(B/6T)^{2/3}$.

When $\omega_1/\Omega_{ce} \gtrsim 1$, the inner electron dynamics is not integrable even in the absence of e_2. However, guiding-center ions may still form whenever e_1 is sufficiently deeply bound that all its dynamical frequencies are much higher than those of e_2. In this case the time-averaged Hamiltonian of Eq. (2) remains valid: the inner electron and the central positive charge act as a neutral

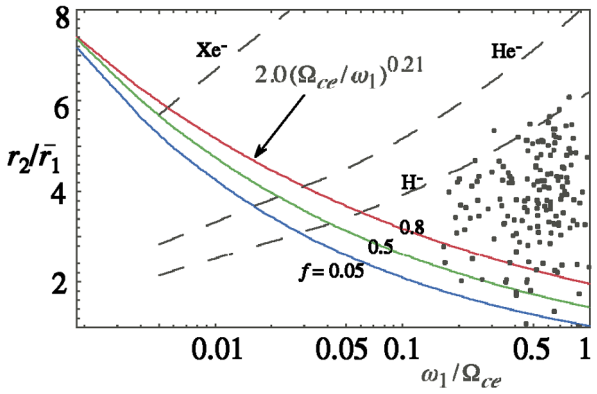

FIG. 3 (color online). Stability diagram for classical electronic guiding-center atoms.

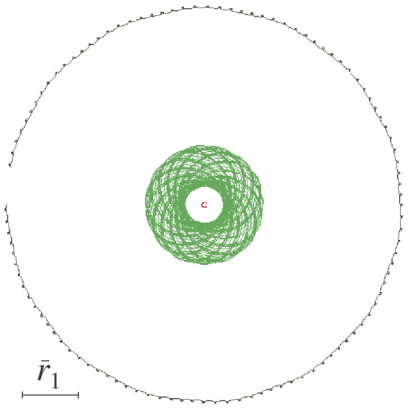

FIG. 4 (color online). Orbits in the xy plane of a guiding center H^- ion with $\omega_1/\Omega_{ce} = 10$. The length scale $\bar{r}_1$ is as indicated.

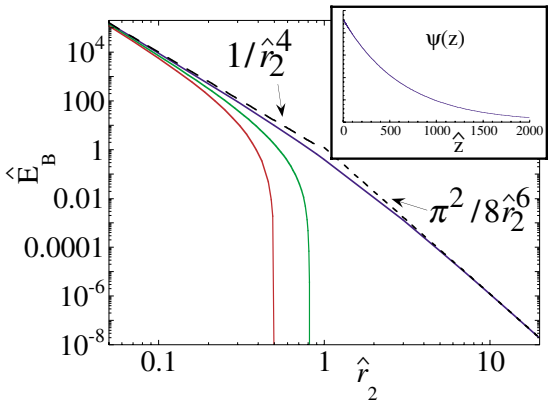

FIG. 5 (color online). Binding energy for the lowest three states of a quantum guiding-center ion. Inset: Ground state eigenfunction, at $r_2/a = 10$.

particle with given quadrupole moment Q and (positive) van der Waals coefficients α_r and α_z. However, the values of these parameters must be determined numerically, and for very tight binding e_1 must be treated quantum mechanically; indeed, e_2 may require a quantum mechanical description as well if it is sufficiently tightly bound, even if its xy motion is well described by guiding-center dynamics. We refer to such ions as quantum guiding-center ions.

We now consider the unusual wave function of quantum guiding-center ions. Assume that $Q = 0$ and $\alpha_r = \alpha_z = R^3$ (i.e., the central neutral particle is spherically symmetric, with radius of order R). Also, assume that $r_2 \gg \sqrt{\hbar/m\Omega_{ce}} \gg R$, where $\sqrt{\hbar/m\Omega_{ce}}$ is the quantum Larmor radius. Then the xy motion of e_2 can be treated classically; i.e., the wave function for e_2 is localized at some given radius r_2. However, the axial motion of e_2 is strongly affected by quantum uncertainty.

Schrödinger's equation for the z motion of e_2 is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(z)}{\partial z^2} - \frac{e^2R^3}{(r_2^2 + z^2)^2}\psi(z) = -E_b\psi(z), \qquad (3)$$

where $\psi(z)$ is the axial wave function. Equation (3) can be put in dimensionless form by scaling distances by the length $a = \sqrt{R^3/a_B}$, and the energy by $\hbar^2/ma^2$, where $a_B = \hbar^2/me^2$ is the Bohr radius. Then Eq. (3) becomes

$$-\frac{1}{2}\frac{\partial^2\psi(\hat{z})}{\partial\hat{z}^2} - \frac{1}{(\hat{r}_2^2 + \hat{z}^2)^2}\psi(\hat{z}) = -\hat{E}_b\psi(\hat{z}), \qquad (4)$$

where the hat denotes the use of scaled variables. The energy levels of the first three excited states are plotted in Fig. 5 versus $\hat{r}_2$, and the lowest wave function is plotted in the inset at $\hat{r}_2 = 10$. The energy levels can be understood analytically in two limiting cases: for $\hat{r}_2 \ll 1$, a Taylor expansion about $\hat{z} = 0$ gives a harmonic oscillator potential, with energy levels $\hat{E}_b = 1/\hat{r}_2^4 - \hat{\omega}_z(n + 1/2)$, where $\hat{\omega}_z^2 = 2/\hat{r}_2^3$. This limit connects to the classical z dynamics discussed previously [since $r_2/a = (r_2/R)\sqrt{a_B/R} \to 0$ in the classical limit].

However, for large $\hat{r}_2$ ($\hat{r}_2 > 0.8165$) only a single bound state exists. Asymptotic analysis of Eq. (4) implies that this energy level is $\hat{E}_b = \pi^2/8\hat{r}_2^6$ (the dashed line in Fig. 5), and the wave function is $\psi = e^{-\sqrt{2\hat{E}_b}|\hat{z}|}$ for $|\hat{z}| \gg 1$. This wave function is highly elongated along the z direction due to quantum uncertainty, with an axial extent of order $a\hat{r}_2^3$. For example, consider an atom or molecule for which $R = 4a_B$ (a reasonable value for the ground state of many common atoms and molecules) and for which $r_2 = 10a = 80a_B$. The outer electron wave function (inset in Fig. 5) then extends to $\hat{z} \sim 1000$, i.e., $z \sim 8000a_B = 0.4\,\mu$m.

Several aspects of these unusual ions remain to be addressed; such as the effect of cyclotron motion on ion stability, formation and reionization rates, and inclusion of radial dynamics in the quantum wave functions. These and other issues will be examined in forthcoming work.

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Note added in proof.—It has come to our attention that related work has considered the binding energy and stability of a central neutral particle in or near the ground state and a weakly bound electron [12]. This work also predicts the large $\hat{r}_2$ limit of the binding energy curve shown in Fig. 5.

[1] H. Goldstein, *Classical Mechanics* (Addison-Wesley, New York, 1950).
[2] D. Lai, Rev. Mod. Phys. **73**, 629 (2001).
[3] B. B. Kadomtsev and V. S. Kudryavtsev, JETP Lett. **13**, 42 (1971).
[4] See, for example, R. O. Mueller *et al.*, Phys. Rev. A **11**, 789 (1975); D. M. Larsen, Phys. Rev. B **23**, 4076 (1981); M. Vincke and D. Baye, J. Phys. B **22**, 2089 (1989); O-A. Al Hujaj and P. Schmelcher, Phys. Rev. A **61**, 063413 (2000).
[5] See, for example, J. C. Lopez *et al.*, Phys. Rev. A **56**, 4496 (1997); C. S. Wurke and A. K. Dutta, Phys. Rev. A **16**, 1747 (1977).
[6] T. F. Gallagher, *Rydberg Atoms* (Cambridge University Press, Cambridge, United Kingdom, 1994).
[7] M. E. Glinsky and T. M. O'Neil, Phys. Fluids B **3**, 1279 (1991).
[8] G. Gabrielse *et al.*, Phys. Rev. Lett. **89**, 233401 (2002).
[9] M. Amoretti *et al.*, Nature (London) **419**, 456 (2002).
[10] S. G. Kuzmin, T. M. O'Neil, and M. E. Glinsky, Phys. Plasmas **11**, 2382 (2004); S. G. Kuzmin and T. M. O'Neil, Phys. Rev. Lett. (to be published).
[11] C. F. Driscoll, Phys. Rev. Lett. **92**, 149303 (2004).
[12] V. G. Bezchastnov *et al.*, Phys. Rev. A **61**, 052512 (2000); V. G. Bezchastnov *et al.*, Phys. Rev. A **65**, 032501 (2002), and references therein.