Phase mixing and echoes in a pure electron plasma\(^a\)

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The two-dimensional (2D) fluid echo is a spontaneous appearance of a diocotron wave after two externally excited waves have damped away, explicitly demonstrating the reversible nature of spatial Landau damping. The inviscid damping, or phase mixing, is directly imaged by a low-noise charge-coupled device camera, which shows the spiral wind-up of the density perturbation. Surprisingly, the basic echo characteristics agree with a simple nonlinear ballistic theory that neglects all collective (i.e., mode) effects. Also, the simple 2D picture is violated by end confinement fields that cause \(v_z\)-dependent \(\theta\) drifts, so the observed echo must be interpreted as a superposition of separately damping and separately echoing velocity classes. The maximal echo lifetimes agree with a theory describing weak collisional velocity scattering between velocity classes. In addition, large second wave excitations degrade the echo up to \(5 \times \) faster than collisions. © 2005 American Institute of Physics. [DOI: 10.1063/1.1885006]

I. INTRODUCTION

Echo phenomena have been seen in a variety of systems spanning many scientific disciplines, including optics,\(^1\) atomic physics,\(^2\) and plasma physics.\(^3–5\) In general, echoes involve the dissipationless phase mixing of a macroscopic signal. Later in time, a second excitation is applied and this perturbation also damps. The echo is the “spontaneous” appearance of a macroscopic signal long after the two external excitations damp away, and provides an explicit demonstration that the phase mixing is thermodynamically reversible.

Here, experimental measurements of two-dimensional (2D) fluid echoes are presented, demonstrating the reversible nature of spatial Landau damping. Fluid echoes have been predicted theoretically,\(^6,7\) and studied numerically,\(^8\) but viscosity or other nonideal effects have prevented echo observations in conventional fluids. We use a magnetized electron column as the “working fluid” to study the echo; the \(z\)-averaged \(\mathbf{E} \times \mathbf{B}\) drift dynamics of the electron column is isomorphic to the 2D \((r, \theta)\) dynamics of an ideal (incompressible and inviscid) fluid.\(^9\) Thus, the echoes represent Kelvin wave\(^10\) echoes, that is, surface wave echoes on a 2D inviscid vortex. We demonstrate that the echo mode number, echo appearance time, and nonlinear saturation effect agree with a simple nonlinear ballistic theory. In addition, we find that the maximal echo lifetime is fundamentally limited by electron-electron collisions, but can also be limited by large amplitude effects.

To produce the echo, we first launch a surface drift wave on a quiescent electron column with density \(n_0(r)\). These “diocotron waves”\(^11\) have density perturbation \(\delta n_{1}(r)e^{i(m_1\theta-\omega t)}\) with \(m_1=2, 3, 4, \ldots\), and are essentially uniform in \(z\). The measured wave electric field at the cylindrical wall is proportional to the radial integral over \(\delta n\). For our chosen profiles \(n(r)\), the excited wave rapidly damps away by the phase mixing of spatial Landau damping;\(^12\) strong radial shear in the \(\mathbf{E} \times \mathbf{B}\) rotation frequency \(\omega_E(r)\) causes progressive spiral wind-up of the perturbation \(\delta n(r, \theta, t)\) near the critical radius \(r_c\), where the wave-particle resonance \(\omega_E(r_c)=\omega_i/m_i\) is satisfied. Essentially, each “shell” of the plasma rotates at a different rate, stretching the initially aligned (\(\theta\) localized) density perturbation into a spiral. The perturbed density then becomes a rapidly oscillating function of \(r\), so the wave potential \(\delta V\) phase mixes to zero. Although the wave potential vanishes, the spiral phase-mixing pattern remains stored in \(\delta n(r, \theta, t)\).

After a chosen time \(\tau\), a second diocotron wave is launched and it too damps away, leaving its own phase-mixing pattern \(\delta n(r, \theta, t)\). The second wave excitation also modulates the perturbation remaining from the first wave, producing a second-order perturbation \(\delta n^{(2)}(r, \theta, t)\). This second-order perturbation begins to unwind, and after some time the perturbations in the various shells realign, producing a wave electric field which is the echo.

A simple representation of spiral wind-up and echo formation is shown in Fig. 1, using initial azimuthal mode number \(m_i=2\) and second mode number \(m_s=4\). Here, white represents positive \(\delta n(r, \theta, t)\), black represents negative \(\delta n(r, \theta, t)\), and \(\delta n\) is treated as a passive tracer density field that is advected by the sheared \(\mathbf{E} \times \mathbf{B}\) flow. The essential feature of the second excitation is to produce radial drifts of particles, shown in Fig. 1(d). The modulated spiral pattern evolves with the shear flow \(\omega_E(r)\), and after some time begins to unwind. At time \(t\tau=2\), the perturbation forms an \(m_i=2\) echo, shown in Fig. 1(f).

This simple description of the 2D echo is complicated by 3D “end” effects that make \(\omega_E\) dependent on an electron’s \(z\) velocity, i.e., \(\omega_E(r, v_z)\).\(^13\) That is, energetic electrons pen-
etrate further into the end confinement potential, and therefore have a different $z$-averaged $\omega_E(r,v)$. Different $z$-velocity classes must separately phase mix and unmix to form a combined echo, so it is somewhat surprising that the echo occurs at all (because these 3D end effects lie outside the 2D fluid perspective). We will see that collisional velocity scatterings between velocity classes fundamentally limit this recombination.

The fluid echoes are closely related to plasma wave echoes. Indeed, the description of a plasma wave echo differs from the above 2D description only in that the wave electric field is given by an integral over the perturbation in the phase-space distribution $d\delta f(z,p_z,t)$. The Landau damping of $k_z \gg 0$ plasma waves is a phase-mixing process where $d\delta f(z,p_z,t)$ becomes progressively more rapidly oscillating in $p_z$, and the integral vanishes. However, a phase-mixing pattern remains stored in $d\delta f(z,p_z,t)$, and this produces an echo when it unmixes after modulation by a second wave.

II. EXPERIMENTAL SETUP

The experiments are performed on a magnetized pure electron plasma confined in a Penning–Malmberg trap, shown in Fig. 2. A hot tungsten source injects electrons into a stack of cylindrical electrodes. The electrons are trapped axially by voltages $-V_c = -100$ V applied to end cylinders; the strong axial magnetic field $0.2$ kG $\ll B \ll 7$ kG provides radial confinement, and causes rapid cyclotron motion at frequency $v_c$. The vacuum chamber (with neutral gas pressure $P \approx 10^{-10}$ torr) resides inside the bore of the superconducting solenoid.

The plasma typically has a central density $n_o \approx 1 \times 10^7$ cm$^{-3}$, with a length varied over the range $5 \approx L_p \approx 70$ cm inside a wall radius $R_w = 3.5$ cm. We adjust the plasma radius to $R_p \approx 1.5$ cm, and create a broad radial tail to produce strong damping, as shown in Fig. 3.

The electrons have thermal energy $4 \approx T \approx 10$ eV, giving an axial bounce frequency $f_b = \bar{v}/2L_p \approx (0.5 \text{ MHz})T^{1/2}(L_p/50)^{-1}$ and a Debye shielding length $\lambda_D \approx (0.25 \text{ cm})T^{1/2}(n/10^7)^{-1/2}$. Here, the thermal velocity is $\bar{v} = \sqrt{T/M}$. The electron space charge creates a central potential $-\phi_p \approx -30$ V, and the radial electric field causes $E \times B$ rotation at a rate $f_E(r) = cE(r)/2\pi rB \approx (150 \text{ kHz})(n/10^7)$.
tron density \( n \) the signal from a wave with a particular mode number \( m \). Receiver, or sectors can be used in combination to maximize waves in the plasma. A single wall sector can be used as applicable, and the fast bounce motion compared with the \( f_E \). The fast cyclotron motion makes guiding-center theories applicable, and the fast bounce motion compared with the \( \mathbf{E} \times \mathbf{B} \) drift frequency makes the system approximately 2D.

Sected wall cylinders are used to excite and detect waves in the plasma. A single wall sector can be used as a receiver, or sectors can be used in combination to maximize the signal from a wave with a particular mode number \( m \).

At a chosen time in the evolution, the \( z \)-integrated electron density \( n(r, \theta, t) \) can be measured by dumping the plasma onto a phosphor screen (biased to 15 kV), imaged by a low-noise 512 \( \times \) 512 charge-coupled device (CCD) camera. The shot-to-shot reproducibility is good \( |\Delta n(r, \theta, t)/n| \approx 0.1\% \), so a time evolution is obtained by creating a sequence of plasmas with identical wave excitations, each dumped at differing times \( t \).

\( \times (B/1 \) kG\( )^{-1} \). Later, in Sec. V, we consider 3D end effects that make the rotation frequency depend slightly on \( v_z \).

The frequency ordering is thus
\[
\Omega_e/2\pi \gg f_B > f_E f_m,
\]
where \( \Omega_e = eB/mc \) is the cyclotron frequency, and \( f_m \) is the frequency of the diocotron mode with azimuthal mode \( m \). The fast cyclotron motion makes guiding-center theories applicable, and the fast bounce motion compared with the \( \mathbf{E} \times \mathbf{B} \) drift frequency makes the system approximately 2D.

FIG. 4. Measured wall signal showing initial, second, and echo waves.

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The corresponding 2D density is initially “circular,” i.e., \( n_0(r) \) but it is distorted into an elliptical \( n(r, \theta) \) by the initial excitation. The peak received initial and echo wall signals are denoted \( S_i \) and \( S_e \), respectively.

The “perturbation” images of Fig. 5 have the symmetric equilibrium subtracted out, displaying \( \Delta n(r, \theta, t) = n(r, \theta, t) - n_0(r) \), where \( n_0(r) \) is the profile obtained at \( t=0 \) with no wall excitation. The colors show magnitude of \( \Delta n \); the initial amplitude is \( |\Delta n| \approx 10^{-2} n_0(r) \) but the color scale is adjusted from image to image to maximize the visibility. The elliptical distortion rotates counterclockwise at frequency \( \omega_B \) but in \( \sim 5 \) wave periods it damps back to a circular cross section due to spatial Landau damping, with spiral wind-up of the density perturbation.

At time \( \tau \), a second wave with \( m_f = 4 \) is excited by applying a voltage \( V_s = 0.2 - 10 \) V to four sectors for \( \Delta \tau \sim \pi/\omega_B \). (The visible wall signal represents a spurious coupling of the \( m_f = 4 \) excitation into the \( m = 2 \) detection elec-

FIG. 5. (Color). Experimental density perturbation images at six successive times. The initial wave is excited in (a) and the density perturbation executes spiral wind-up as the wave is spatially Landau damped. (d) shows the second wave excitation and the echo peaks in (f).

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tronics.) The remnants of the phase-mixed initial wave are visible as thin filaments in (d). The second wave excitation causes $\theta$-dependent radial shifts of the filaments; thus, the filaments $E \times B$ rotate at a new rate because $\omega_E(r)$ depends on $r$.

The third wave packet visible in Fig. 4 is the received $m_s=2$ echo. The initial wave damping effectively unmixed, producing an $m_s=2$ echo response. The peak echo wall signal $S_e$ occurs at a time $t=2\tau$ for the mode numbers used here. The corresponding image (f) shows $|\delta n|=(0.3 \times 10^{-2})n(r)$.

### IV. COLLISIONLESS BALLISTIC THEORY

Surprisingly, the essential features of the echo are captured by a collisionless, ballistic theory\(^4\) that completely neglects collective effects such as waves. We solve for the free-streaming $E \times B$ drift trajectories of particles, and assume that particle-guiding centers deviate from circular orbits only due to the impulsively applied wall excitation voltages. This ballistic approach is equivalent to treating the particles as passive tracers that orbit at rate $\omega_E(r)$; that is, we assume the perturbed density does not modify the $E \times B$ flow.

Here, we calculate the electric field $E_w$ at the wall of the electron trap. The $m$th Fourier component of the electric potential is

$$\delta \phi_m(r,t) = -e \int_0^{R_w} r'dr' \int_0^{2\pi} d\theta' n(r',\theta',t)G_m(r|r')e^{-i\omega t},$$

where $(r', \theta')$ is the source point and $G_m(r|r')$ is the Green function.\(^{14}\)

Following Ref. 4, we use conservation of particles

$$r'dr'd\theta' n(r',\theta',t) = r_d dr_d d\theta_d n_o(r_o)$$

to express the particle trajectories in terms of the initial $(t=0)$ coordinates $(r_o, \theta_o)$ and unperturbed density $n_o(r_o)$. The electric field at the wall is then

$$E_w = - \frac{\partial \delta \phi_m(r)}{\partial r} \bigg|_{r=R_w}$$

$$= e \int_0^{R_w} r_d dr_d \int_0^{2\pi} d\theta_d n_o(r_o) \frac{\partial}{\partial r} \left[ G_m(r|r',\theta_o,t) \right] e^{-i\omega t},$$

where we drop the prime on $\theta$ for notational convenience and the radial derivative of the Green function, evaluated at $r=R_w$, is

$$g_m(r|r') = \frac{\partial G_m(r|r')}{\partial r} \bigg|_{r=R_w} = \omega_m^{m-1}.$$  

Below, we solve for the particle trajectories described by $\theta(r_o, \theta_o, t)$ in the exponential argument of Eq. (4).

The Hamiltonian for 2D $E \times B$ drift dynamics is given by

$$H = -e\phi_0(r) + \delta H,$$

where $\phi_0(r)$ is the unperturbed plasma potential. In general, the perturbed Hamiltonian $\delta H$ would include self-consistent plasma effects; these effects are ignored in this ballistic approximation. The perturbed Hamiltonian models vacuum potentials created by wall voltages $V_r$ and $V_o$, applied impulsively at $t=0$ and $t=\tau$,

$$\delta H = eA_i(r)\sin(m_i\theta)\delta(t) - eA_o(r)\sin(m_o\theta)\delta(t-\tau),$$

where $A_i(r) = a_i V_i \Delta t_i (r/R_w)^{m_i}$, $A_o(r) = a_o V_o \Delta t_o (r/R_w)^{m_o}$ represent the $r$ dependence of the vacuum potentials from the initial and second excitations, and $\Delta t_i$, $\Delta t_o$ represent the actual durations of the applied wall voltages. Here $(a_i, a_o)$ relate the voltages applied on wall sectors to the $(m_i, m_o)$ spatial Fourier components.

The equations of motion follow from Eqs. (6) and (7):

$$\frac{\partial \theta}{\partial t} = \frac{\partial H}{\partial \phi_0} = \frac{c}{eB r} \frac{\partial \phi_0}{\partial r} = \omega_E(r) + \frac{c}{Br} \left[ \frac{\partial \Lambda_i}{\partial r} \sin(m_i\theta) \delta(t) \right. + \frac{\partial \Lambda_o}{\partial r} \sin(m_o\theta) \delta(t-\tau) \right],$$

$$\frac{\partial r}{\partial t} = \frac{c}{eB r} \frac{\partial \phi_0}{\partial \theta} = -\frac{c}{eB r} \frac{\partial H}{\partial \theta} = \frac{c}{Br} \left[ A_i(r) \cos(m_i\theta) \delta(t) \right. - \left. A_o(r) \cos(m_o\theta) \delta(t-\tau) \right],$$

where $\omega_E(r) = -c/l_B \partial \phi_0/\partial r$ is the 2D $E \times B$ rotation rate, and we use $p_0=(eB/2c)r^2$ to express the canonically conjugate variables $(\theta, p_\theta)$ as $(\theta, r^2)$. Integrating the equations of motion from $t=0$ to a time $t<\tau$, before the second wall excitation, gives

$$\theta = \theta_o + \omega_E(r)t + \delta\theta(\theta_o), \quad 0 < t < \tau$$

$$r^2 = r_o^2 + A_i(r) m_i \frac{2c}{B} \cos(m_i\theta_o), \quad 0 < t < \tau,$$

where $\delta\theta = c/l_B \partial A_i/\partial \theta \sin(m_i\theta_o)$ is the impulsive $\theta$ drift from the radial electric field of the initial excitation. These $\theta$ kicks are ignored in the remaining theory because they are not multiplied by $t$, so they do not accumulate with time.

Integrating Eqs. (8) and (9) from $t=\tau$ to a time $t>\tau$, after the second wall excitation, i.e., when the echo appears, gives

$$\theta = \theta(\tau) + \omega_E(r)\tau + \delta\theta(\theta(\tau)), \quad \tau > \tau$$

$$r^2 = r_o^2 + A_i(r) m_i \frac{2c}{B} \cos(m_i\theta(\tau)) - A_o(r) m_o \frac{2c}{B} \cos(m_o\theta(\tau)), \quad t > \tau,$$

where $t=\tau'$ is the time immediately after the second excitation. The radial displacements caused by the second wall excitation are given by Eq. (13). Particles displaced to a new radial position $r(\tau')$ rotate at the new rate $\omega_E(r(\tau'))$, enabling unmixing.

Using Eqs. (10) and (11), we evaluate $\theta(\tau)$ as
\[ \theta(\tau) = \theta_o + \omega_E \left[ r_o + \frac{c}{B_r} m_i A_i(r) \cos(m_i \theta_o) \right] \tau \]

\[ \approx \theta_o + \omega_E(r_o) \tau + \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} \bigg|_{r_o} A_i(r) m_i \cos(m_i \theta_o) \tau, \]

(14)

where \( \omega_E \) is Taylor expanded in the last line. Similarly, the term \( \omega_E[r(\tau^2)] \) in Eq. (12) is Taylor expanded. The particle \( \theta \) trajectories of Eq. (12) are then

\[ \theta(r_o, \theta_o, t) = \theta_o + \omega_E(r_o) t + \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} \bigg|_{r_o} A_i(r) m_i \cos(m_i \theta_o) t \]

\[ - (t - \tau) \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} \bigg|_{r_o} A_i(r) m_i \]

\[ \times \cos \left[ m_i \theta_o + m_o \omega_E(r_o) \tau \right] \]

\[ + \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} \bigg|_{r_o} A_i(r) m_i m_e \cos(m_i \theta_o) \tau \bigg] \right]. \]  

(15)

We make use of the exponential expansion

\[ e^{-i a \cos s} = \sum_i (-i)^i J_i(a)e^{-i l x}, \]

(16)

where \( J_i \) is the Bessel function of the first kind of order \( l \), and find that the electric field at the wall vanishes unless \( m = q m_s - p m_i \). Here, \( p \) and \( q \) are the harmonic orders of the initial and second excitations, respectively, similar to \( l \) in Eq. (16).

For the lowest harmonic numbers \( p \equiv q = 1 \), the echo mode number is predicted to be

\[ m_i = m_s - m_e. \]

(17)

Experiments with a variety of \( (m_i, m_s) \) show \( m_i = m_s - m_e \) and no echo is seen if \( m_i > m_s \).

Furthermore, we use the Bessel function identity

\[ J_p(c - d) = \sum_i J_{p+i}(c) J_i(d), \]

(18)

and find that the electric field at the wall, for \( t > \tau \), is given by

\[ E_w(t) = e^{i \int_0^{r_w} rdT_0(r') \rho_0(r') \rho_0(r') \exp \left[ -im_s \omega_E(r) \right] \}

\[ \times \left( t - \frac{m_i}{m_s} \right) J_1 \left( \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} A_i(r) m_i m_e \right) \]

\[ \times \left( t - \frac{m_i}{m_s} \right) J_1 \left( \frac{c}{B_r} \frac{\partial \omega_E}{\partial r} A_i(r) m_i m_e (t - \tau) \right) \]. \]

(19)

where we have dropped the subscript on \( r \) for notational convenience.

A. Time of echo appearance

Equation (19) predicts the time of the echo appearance, which is obtained by setting the phase-mixing terms to zero. In the experiments, phase mixing is the spiral wind-up of the density perturbation, creating radial ripples of the \( \delta n \) which enters the theory. Figure 6 shows the measured \( t_e \) versus the second wave launch time \( \tau \), both normalized to the \( m = 2 \) wave period \( T_2 \), using various mode numbers.

![Figure 6](image)

**FIG. 6.** Echo appearance time \( t_e \) vs the second wave launch time \( \tau \), both normalized to the \( m = 2 \) wave period \( T_2 \), using various mode numbers.

We measure the echo appearance time \( t_e \) with an automated fit to the wall signal, using a symmetric growing/damping sine wave centered at \( t_e \). The sine wave frequency is at the well-defined mode frequency, which does not even enter the theory. Figure 6 shows the measured \( t_e \) versus the second wave launch time \( \tau \), both normalized to the \( m = 2 \) wave period \( T_2 \). The dashed lines are the theory predictions, and the initial, second, and echo mode numbers are shown in parentheses. The case with \( m_i = 2 \) and \( m_s = 4 \) has the largest data range due to the large signal-to-noise ratio of the measured echo amplitude.

To further compare the collisionless theory with experiments, we evaluate Eq. (19) near the echo appearance time \( t_e \), and write the “no collision” theory prediction for the peak echo wall signal as

\[ S_e^c = G E_w(t = t_e) = V_i \int_0^{r_w} dr \alpha(r) J_1[\beta(r) V_i], \]

(21)

where \( \alpha(r) \) and \( \beta(r) \) are given by

\[ \alpha(r) = G a \Delta t \frac{\epsilon c m_i}{2B r} \left( r \frac{m_i}{m_s} \right) \frac{m_i}{m_s} \left( \frac{m_i}{m_s} \right) \frac{\partial \omega_E}{\partial r}, \]

(22)

\[ \beta(r) = a \Delta t m_i \frac{c}{B r} \left( \frac{m_i}{m_s} \right) \frac{m_i}{m_s} \frac{\partial \omega_E}{\partial r}. \]

(23)

The gain is given by \( G = (A/C) \tilde{G} \), where \( A \) is the area and \( C \) is the capacitance of the detection sectors, and where \( \tilde{G} \) is the amplifier gain. The small quantity \( \epsilon \) represents the difference.
between the theoretical echo appearance time \( t_e \) and the measurement time. This measurement time is defined as the time of the maximal echo wall signal, and is typically within 1 or 2 plasma rotation periods of the theoretical \( t_e \).

Equation (21) follows from Eq. (19) because at \( t=\tau \), the Bessel function argument in Eq. (19) containing the initial perturbation amplitude is extremely small, and \( J_1(x) \approx x/2 \) for small \( x \). The Bessel function argument which contains the second perturbation amplitude is large at the time of the echo appearance, so this Bessel function is not expanded. Theoretically, at \( t=\tau \), the echo wall signal is zero, since \( J_1(0) = 0 \).

V. COLLISIONAL IRREVERSIBILITY OF END-FIELD \( \theta \) DRIFTS

In this section, we evaluate a second-order theory that describes collisional scattering between \( v_z \) velocities, giving irreversible "\( \theta \) smearing." \( \theta \) motion outside the 2D perspective arises from the radial component of the confinement fields, which create \( \mathbf{E} \times \mathbf{B} \) drifts in the \( \theta \) direction.\(^{13}\) These \( \theta \) steps depend on the particle’s end-residence time, and hence on the particle’s \( z \) velocity \( v_z \), so one must consider \( \omega_k(r,v_z) \).

In the absence of collisions or other randomizing effects, each particle remains in its original velocity class, which executes its own spiral wind-up (and unwinding). Thus, the spiral wind-up due to \( \omega_k(r) \) and the wind-up due to \( \omega_k(v_z) \) have similar phase-mixing effects, and the \( v_z \)-dependent \( \theta \) steps alone do not destroy the echo.

Collisions, however, cause irreversible velocity mixing. Particles scattered into a new velocity class change their bounce-averaged rotation rate \( \omega_k(r,v_z) \), which degrades the coherence of the spiral “unwinding,” and limits the maximal echo lifetime.

A. Collisional theory

The Boltzmann equation is used to model collisions acting on the particle distribution function \( f(r,v_z,t) \),

\[
\frac{\partial f}{\partial t} + [f,H'] = v_z^2 \frac{\partial f}{\partial v_z}.
\]

(24)

where \( \nu \) represents the electron-electron parallel scattering rate \( \nu_1 \),\(^{15}\) given by

\[
\nu_1 = 2.8 \sqrt{\pi m_e b^2} \ln(r_L/b).
\]

(25)

Here, \( r_L = v/L \) is the cyclotron radius and \( b = e^2/T \) is the distance of closest approach for thermal electrons.

The Hamiltonian \( H' \) used here includes \( z \) dynamics, and is given by

\[
H' = \frac{p_z^2}{2M} - e\phi(r,z) + \delta H'.
\]

(26)

The perturbed Hamiltonian \( \delta H' \) neglects collective effects, similar to Eq. (7), and models the impulsively applied wall excitations:

\[
\delta H' = eA_1(r)e^{-i\beta t} \delta(t) + eA_1(r)e^{i\beta t} \delta(t-\tau).
\]

(27)

For \( t<\tau \), the first-order solution from the initial excitation only is

\[
\delta f_1^{(1)} = -i m_A eB r \frac{\partial M}{\partial \beta} \exp \left[ -i m_\gamma + \frac{1}{3} \frac{4 m_e^2 v_z^2}{\Omega_L \xi r^3} \left( \frac{\partial L}{\partial t} \right)^2 \right] .
\]

(28)

The \( \omega_k(r,v_z) \) effect is seen in the third term in the exponential argument. Here, \( \frac{\partial M}{\partial \beta} \) is the radial derivative of the unperturbed distribution function, which is assumed to be Maxwellian, and \( \partial L/\partial t \) represents the curvature in the ends of the plasma. In obtaining Eq. (28), we have made the approximation \( \frac{\partial^2 \hat{r}}{\partial t^2} \Delta y = 1/\delta(\partial / \partial v_z) \approx 2 \). The rather strange \( \tau \) scaling of the collisional term in the exponent comes from the collision operator \( \nu e^2 \partial / \partial v_z \) acting on \( \delta H' \), which has the \( v_z \)-dependent \( \theta \) smearing in \( \omega_k(r,v_z) \). This brings down \( \tau \) and the integral \( \nu e^2 \partial / \partial v_z \) \( \Delta y \) then yields \( \tau \).

Solving the Boltzmann equation to second order, and keeping only terms containing the first power of \( \nu \), we find that the perturbed distribution function at \( t>\tau \) is given by

\[
\delta f_2^{(2)}(t) = i m^2_A eB r \frac{\partial M}{\partial \beta} \exp \left[ i m_\gamma t + \frac{1}{3} \frac{4 m_e^2 v_z^2}{\Omega_L \xi r^3} \left( \frac{\partial L}{\partial t} \right)^2 \right] .
\]

(29)

Again, the time of the echo appearance \( t_e \) is obtained by setting the \( r \)-dependent (phase mixing) terms in the first exponential argument equal to zero.

At time \( t=t_e = m_e / m_e \), the second-order collisional solution for the peak echo amplitude is

\[
\delta f_2^{(2)}(t_e) = \exp(i m_\gamma t) \exp(-\gamma^2(v,r,e) \tau^2).
\]

(30)

where the collisional damping term is given by

\[
\gamma^2(v,r,e) = \nu e \left( \frac{c}{eB} \frac{\partial L}{\partial t} \right)^2 \frac{4 m_e^2 v_z^2}{3 m_e}.
\]

(31)

Here, \( e_e = M v_e^2 / 2 \) is the electron kinetic energy, and the plasma length \( L(r,e) \) is defined by the electron end reflection at \( L/2 \):

\[
e \phi(r, \pm L/2) - e \phi(r, 0) = e_e.
\]

(32)

Finally, we combine the collisionless result of Eq. (21) with the collisional result of Eq. (29), and find that the peak echo wall signal, near the echo appearance time, is given by
FIG. 7. Measured peak echo wall signal $S_e$ vs the second wave launch time $\tau$ for two different second excitation voltages $V_s$. The solid lines show fits from numerically integrating the collisional theory expression of Eq. (33). The dashed and dot-dashed lines have $\nu_{\text{eff}}=0$.

\[
S_e^{\text{col}} = \alpha' V_s \int_0^{R_w} dr \left( I_0 \left( \frac{\gamma_0}{r} \right) + \frac{\gamma_0}{r} J_1 \left( \frac{\gamma_0}{r} \right) \right) \left( e^{\nu \tau} \right)
\times \frac{1}{2\sqrt{\pi T_j}} \int_0^{\infty} \frac{dz}{\sqrt{\pi}} \exp \left[ - \frac{e^2}{4z} - \frac{1}{2} \left( \nu_{\text{eff}} \tau + e^2 \right) \right],
\]

(33)

where $\alpha'$ and $\beta'$ are the coefficients of $\alpha$ and $\beta$ in Eqs. (22) and (23), and are given by:

\[
\alpha' = G a_2 M_2 e c m_m e / 2B R_w^{m+1},
\]

(34)

\[
\beta' = a_2 m_m c / B R_w^{m+1}.
\]

(35)

The $1/\sqrt{\pi}$ term arises from changing a velocity integral $\delta n = \int dr e^2 dt$ to an energy integral $\delta n = \int dE e^2 / \sqrt{2M_e} \pi$. In order to compare with experiments, the collision rate $\nu$ in the collisional damping term $\gamma^0$ has been written in terms of an effective collision rate $\nu_{\text{eff}}$:

\[
\nu = \nu_{\text{eff}} n(r) (e^{\nu_{\text{eff}} \tau})^{-1}.
\]

(36)

B. Comparison with experiments

We numerically evaluate the double integral in Eq. (33). A 2D Poisson solver is used to calculate the local density $n(r,z)$ and potential $\phi(r,z)$ from the measured $z$-integrated density from a CCD image. In obtaining the solution, local thermal equilibrium is presumed at every $r$ and $z$. The plasma length $L(r,e)$ is then calculated from Eq. (32), and the plasma rotation frequency is calculated from $\omega_k(r) = -c / B r \phi(r) / \partial r$, at $z=0$.

We obtain an effective electron-electron collision rate $\nu_{\text{eff}}$ by fitting $S_e^{\text{col}}$ of Eq. (33) to data sets of the measured peak echo wall signal $S_e$ versus $\tau$, with $\nu_{\text{eff}}$, $\alpha'$, and $\beta'$ as fitting parameters. The solid lines of Fig. 7 are examples of such a fit using different excitation voltages $V_s$; all other parameters are held constant. For the weak second wave excitation of $V_s=0.25$ V, the fit yields $\nu_{\text{eff}}=31$ sec$^{-1}$, which is fairly close to the actual electron-electron scattering rate of $\nu_i=21$ sec$^{-1}$. Thus, the decrease in $S_e$ represents collisional irreversibility. The simple collisionless theory ($\nu_{\text{eff}}=0$) is also shown dashed and dot dashed, using the same $\alpha'$ and $\beta'$ as the solid lines.

At the larger second excitation of $V_s=1.0$ V, the decrease in $S_e$ at $\tau=20T_2$ represents the nonlinear saturation effect described by the Bessel function in Eqs. (21) and (33). Here, the fit yields $\nu_{\text{eff}}=122$ sec$^{-1}$, which does not represent a measure of collisionality, but rather represents the absence of the late-time echo recurrence at $\tau=85T_2$ (and $\tau=150T_2$).

Figure 8 summarizes the comparison between the collisional theory and echo data. The vertical axis is the experimentally determined $\nu_{\text{eff}}$, obtained from fits to 65 data sets such as Fig. 7. The horizontal axis is the actual electron-electron collision rate $\nu_i$ for the specific plasma parameters, given by Eq. (25). The measured $\nu_{\text{eff}}$ is always greater than that expected from $\nu_i$ alone.

In Fig. 8, the strength of the applied wall voltage $V_s$ used to excite the second wave is represented (logarithmically) by the size of the data symbol, with larger symbols corresponding to larger second excitation strengths. A given value of $\nu_i$, identical symbols of different sizes correspond to identical plasma parameters, but with different values of the second wave excitation strength. The strength of $V_s$ is varied over a factor of 50 in the data plotted here. The arrows identify the data points corresponding to Fig. 7.

Figure 8 shows that the echo lifetime is limited by collisional irreversibility of end-field $\theta$ smearing when the second wave excitation is sufficiently small. At large second wave amplitudes, a different (unknown) effect destroys the echo as effectively as collisions enhanced by 100X. These large values of $\nu_{\text{eff}}$ represent the absence of echo recurrence at large values of $V_s$.

Note that the large variations in $\nu_{\text{eff}}$ translate into much smaller variations in the time of echo destruction, because of the $\tau^3$ scaling in the collisional exponential decay argument in Eq. (33). Thus, a factor of 100 enhancement in $\nu_{\text{eff}}$ gives only a 4.6X reduction in the echo viability time.
VI. LARGE AMPLITUDE EFFECT

The absence of echo recurrence at large second wave amplitudes may be due to collective plasma effects, including particle trapping by the wave potential. This trapping effect causes particles to deviate from ballistic, free-streaming orbits, and limits phase unmixing.

Figure 9 shows that the peak measured echo signal $S_e$ is linearly proportional to the initial excitation voltage $V_i$, as predicted by Eq. (33), until $V_i \geq 2$ V. As the excitation amplitude is increased, nonlinear bouncing oscillations are observed in the wall signal of the initial wave damping, indicating that particles are trapped by the initial wave potential.

Presumably, large second wave excitations also cause particle trapping by the second wave potential, although sector geometry prevents direct observation of trapping oscillations. This particle trapping may be responsible for the observed increase in $v_{\text{eff}}$ at large $V_i$ in Fig. 8.

VII. SUMMARY

We have observed fluid echoes in a pure electron plasma, explicitly demonstrating the reversible nature of spatial Landau damping. The phase mixing and unmixing associated with wave damping and echo generation are imaged directly, and the echo mode number, appearance time, and saturation effect agree with a simple ballistic theory. The pure electron plasma behaves like an ideal 2D fluid, despite 3D end effects that make $\omega_E$ dependent on an electron’s $z$ velocity. Different velocity classes separately phase mix and unmix, surprisingly forming the same echo. At late times the echo is degraded, and collisional scattering between velocity classes gives a fundamental limit to the echo lifetime. In addition, large amplitude effects degrade the echo up to 5 times faster than collisions.

9H. Lamb, Hydrodynamics, 6th ed. (Dover, New York, 1932), Secs. 158 and 159; L. Kelvin, Philos. Mag. 10, 155 (1880).
14See National Technical Information Service Document No. DE97053509 [direct numerical solution of Poisson’s equation in cylindrical $(r, \theta)$ coordinates]. Copies may be ordered from the National Technical Information Service, Springfield, VA 22161.