Electron acoustic waves in finite length plasmas: theory and numerical Vlasov simulations

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Abstract

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I. THE THEORY

A. Dispersion relation

We investigate the effects of finite radius on the dispersion relation of the Electron acoustic waves (EAW). We look for solutions of the Vlasov-Poisson equations for a nonneutral electron plasma confined in a cylindrical Penning-Malmberg trap. In cylindrical coordinates \((r, \theta, z)\), the Vlasov-Poisson system is written as follows:

\[
\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \frac{e}{m} \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial v_z} = 0
\]

(1)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi en
\]

(2)

where \(f = f(r, z, v_z, t)\) is the electron distribution function, \(\phi = \phi(r, z, t)\) the electrostatic potential and \(n = n(r, z, t)\) the particle density.

In the previous system we have neglected the \(\theta\)-dependence and the radial motion of the particles, assuming that the large confining magnetic field forces the particle motion along the \(z\) axis. Then, we look for solution in the form:

\[
\delta f(r, z, v_z, t) = \hat{\delta f}(r, v_z) \exp \left[ i(k_z z - \omega t) \right]
\]

\[
\delta \phi(r, z, t) = \hat{\delta \phi}(r) \exp \left[ i(k_z z - \omega t) \right]
\]

In the previous equations, \(k_z = \pi / L_z\) is the wave-number of a plasma column of length \(L_z\) and \(\omega\) is the complex frequency \((\omega = \omega_r + i\gamma)\).

By linearizing the Vlasov equation (1) in the perturbation amplitudes, we get the expression for the perturbed distribution function \(\delta \hat{f}\):

\[
\delta \hat{f}(r, v_z) = \left[ \frac{e}{m} k_z \delta \phi(r) \frac{\partial f_0(r, v_z)}{\partial v_z} \right] (\omega - k_z v_z)^{-1}
\]

(3)

In the above equation, \(f_0(r, v_z)\) represents the equilibrium distribution function, that can be expressed in the form \(f_0(r, v_z) = n(r)\tilde{f}_0(v_z)\), where \(n(r)\) is the radial density profile and \(\tilde{f}_0(v_z) = (1/\sqrt{2\pi v^3}) \exp \left[ -(v^2/2v^3) \right] \) is a maxwellian function in \(v_z\) (the radial dependence in the thermal velocity \(v\) will be taken into account in the following). Using Eq. (3), the linearized Poisson equation reads:

\[
\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - k^2_z + \frac{n(r)}{n(0)} \omega_r^2 \int_L dv_z \frac{\partial \tilde{f}_0}{\partial v_z} \left( v_z - \omega / k_z \right) \delta \phi(r) = 0
\]

(4)
The electron plasma frequency $\omega_p$ is defined with the value of the density at $r = 0$ $n(0)$. The subscript $L$ in the integral over the velocities in the above equation indicates the Landau contour [1]. For sufficiently weak damping, the velocity integral along the Landau contour can be approximated by

$$
\int \frac{d\tilde{f}_0}{v_z - \omega/k_z} = P \int_{-\infty}^{+\infty} \frac{d\tilde{f}_0}{v_z - v_\phi} + \frac{\pi i}{v_\phi} \left. \frac{\partial \tilde{f}_0}{\partial v_z} \right|_{v_\phi},
$$

where $P$ indicates that the Cauchy principle part is to be taken and $v_\phi = \omega_r/k_z$ is the wave phase velocity. The trapped particle distribution for an EAW effectively makes the distribution flat at the wave phase velocity (i.e., $\partial \tilde{f}_0/\partial v_z |_{v_\phi} \simeq 0$). Thus, Holloway and Dorning [2] obtain a dispersion relation for small amplitude EAWs for infinite plasmas, by retaining only the principle part in the velocity integral. Then, for a finite length plasma, the Poisson equation becomes:

$$
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + k_z^2 \frac{n(r)}{n(0)} \right] \hat{\phi}(r) = 0
$$

where we have introduced the quantity $K^2$, defined as:

$$
K^2 = \omega_p^2 P \int \frac{d\tilde{f}_0}{v_z - v_\phi}
$$

If one considers a step function density profile in solving Eq. (6)

$$
n(r) \overline{n(0)} = \begin{cases} 
1, & r \leq R_p \\
0, & R_p < x \leq R_w 
\end{cases}
$$

one obtains two equations for the perturbed electrostatic potential, one inside the plasma column ($r \leq R_p$) and the second one in vacuum ($R_p < r \leq R_w$), where $R_p$ is the plasma radius and $R_w$ is the radial coordinate of the wall.

$$
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + K_{\perp}^2 \right] \hat{\phi}_1(r) = 0, \quad r \leq R_p
$$

$$
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - k_z^2 \right] \hat{\phi}_2(r) = 0, \quad R_p < r < R_w
$$

where $K_{\perp}^2 = K^2 - k_z^2$. The solutions of the previous equations are then obtained as:

$$
\hat{\phi}_1(r) = J_0(K_{\perp} r), \quad r \leq R_p
$$

$$
\hat{\phi}_2(r) = \hat{\phi}_2(R_w) \ln(R_w/r), \quad R_p < r < R_w
$$
where $J_0$ is the Bessel function of zero-th order. In solving the equation for $\delta \phi_2$, we have considered the approximation $k_z r \ll 1$. Imposing the solutions and their first radial derivatives to match at $r = R_p$, we obtain the equation for $K_\perp$, in terms of the Bessel function $J_0[K_\perp R_p]$ and its derivative $J'_0[K_\perp R_p]$:

$$R_p \frac{K_\perp J'_0[K_\perp R_p]}{J_0[K_\perp R_p]} = [\ln(R_p/R_w)]^{-1}$$

Equation (13) can be solved numerically for $K_\perp$, using the characteristic parameters of the plasma column, $R_p \simeq 6.46 \lambda_D$, $R_w \simeq 18.84 \lambda_D$, $L_z \simeq 258.34 \lambda_D$, where $\lambda_D \simeq 0.1858 cm$ is the typical electron Debye length, obtaining the value $K_\perp \simeq 0.19$. The solution for $K^2 = K_{\perp}^2 + k_z^2$ is used to evaluated numerically the wave phase velocity $v_\phi$ from Eq. (7),

FIG. 1: From "thumb" to "finger", for $K_\perp = 0, 0.07, 0.2, 0.25, 0.35, 0.45$. 
giving the value $v_\phi \simeq 1.356\pi$. It is worth noting that the effect of finite radius produces a shift in the phase velocity of the Electron acoustic waves, with respect to the case of infinite plasmas ($R_p \to \infty$), where the solution of the dispersion relation gives a wave phase velocity $v_\phi = 1.31\pi$.

The parameter $K_\perp$ is strictly related to the confining geometry of the trap. One can try to solve the dispersion relation (7), for different values of $K_\perp$, i.e. for different confining geometry characteristic parameters. In Fig. 1, we show the solution of dispersion relation (7), for six different values of $K_\perp$. For $K_\perp = 0$ (top left in the figure), the ”thumb” dispersion relation is obviously found; this solution corresponds to the infinite length plasma EAW dispersion relation. On the other hand, we see that as $K_\perp$ increases, the TG modes appear and the ”thumb” becomes a ”finger”, disappearing if $K_\perp \geq 0.5$. This suggests that in finite geometry, there might be two nonlinear EAWs with very similar phase velocities.

The next step is to find the solution for the Poisson equation (6), considering the real density profile for electrons confined in the cylindrical trap. The radial dependence of the density is shown in Fig. 2. Let us consider the eigenvalue problem defined by the Poisson equation (6) with the following boundary conditions for the perturbed potential:
We solve numerically the above equation through a standard shooting method, using the characteristic plasma parameters listed above and the density profile shown in Fig. 2. We find the solution for $K^2 \approx 0.0451$, that we use to evaluate the phase velocity of the wave through definition (7) and get $v_\phi = 1.369\pi$. As it is easily seen, the value of the phase velocity is slightly different with respect to the case of the step function density profile, but it is still shifted upward with respect to the case of infinite length plasmas. The numerical solutions for the potential eigen function and the absolute value of the density are displayed in Fig. 3; the dashed lines in the figure represent the plasma radius $R_p$ and coordinate of the wall $R_w$ respectively. The density eigen function approaches zero right after the plasma radius and remains zero for $r \to R_w$. 

$$\begin{align*}
\left[\frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + k_z^2 \left(\frac{n(r)}{n(0)} \frac{K^2}{k_z^2} - 1\right)\right] \delta \hat{\phi}(r) &= 0 \\
\delta \hat{\phi}(r = R_w) &= 0 \\
\delta \hat{\phi}(r = 0) &= 1 \\
\partial \delta \hat{\phi}(r)/\partial r |_{r=0} &= 0
\end{align*}$$

FIG. 3: Electrostatic potential and density eigen functions.
Considering the real density profile, we can evaluate \( K_\perp = \sqrt{K^2 - k_z^2} \simeq 0.212 \) (which is slightly different from the value \( K_\perp \simeq 0.19 \) we obtained in the case of step function density profile). Figure 4 shows the comparison between the experimental data for the frequency of the TG modes and the EAWs (the dots in the figure) and the theoretical solution of the dispersion relation (7), for \( K_\perp \simeq 0.212 \) and for the real density profile shown in Fig 2. As it can easily be seen, the results are in very good agreement, in correspondence of the value of the wave number for our experiment \( (k_z \simeq 0.012) \).

B. Radial variation of the plasma temperature

Figure 5 shows the experimental result for the absolute value of the density eigen function (solid line in the plot). The dashed line in the figure represents the plasma radius \( R_p \). Comparing this plot with the theoretical result for the density eigen function in Fig. 3 (at the bottom), it is easily seen that a significative peak appears in the experimental results, that is not visible in the theoretical solution we showed in the previous section. The dotted-dashed line in Fig. 5 represents the phase of the eigen function \( \delta n \); this phase shows that \( \delta n \) crosses zero before the plasma radius \( (r < R_p) \) and changes sign in the region around \( r \simeq R_p \). In the theoretical solution shown in Fig. 3 no evidence appears of this change of sign. The suggestion that this phenomenon could be due to the radial dependence of the temperature
of the plasma column, has been obtained empirically, by varying the temperature of the plasma at the radial edge of the trap. The amplitude of the peak, as well as its radial position, are strongly affected by the value of the temperature at \( r \approx R_w \).

In order to take into account this effect in our theory, we consider the radial dependence of the particle thermal velocity, in solving the eigenvalue problem discussed earlier. The dependence of temperature on the radial coordinate is taken to be \( T(r) = T(0)[1+\alpha(r/R_w)^2] \), where \( T(0) \) is the temperature at \( r = 0 \) (\( T(0) \approx 1 \text{eV} \), for the electrons confined in the trap). The numerical results for the density eigen function are shown in Fig. 6, for four different values of the parameter \( \alpha \). It is clear from the figure that increasing \( \alpha \), i.e. increasing the temperature of the plasma at the radial edges of the trap, the density eigen function goes to zero before the plasma radius \( R_p \) (indicated by dashed lines in the figure). In other words, the density becomes narrower and narrower for higher and higher temperature and changes sign before approaching again zero for \( r \to R_w \). Moreover, the amplitude of the opposite sign peak around \( r \approx R_p \) increases as \( \alpha \) increases.

C. Frequency shift for finite amplitude perturbations

We emphasize that the method used in the previous section to solve the dispersion relation describes only small amplitude EAWs. Using a Maxwellian distribution for \( \tilde{f}_0(v_z) \) and
taking the principle value in the velocity integral assumes that the width of the plateau where \( \partial \tilde{f}_0 / \partial v_z = 0 \) is infinitesimal. For a finite amplitude EAW, the plateau width is the velocity range over which electrons are trapped in the wave troughs, that is \( \Delta v_{\text{trap}} \), where \( (\Delta v_{\text{trap}})^2 \sim e\phi/m \). An infinitesimal trapping width corresponds to an infinitesimal wave amplitude.

In order to take into account the effect of finite amplitude on the dispersion relation of the EAWs in finite radius plasmas, we solve numerically the dispersion relation (7), for a distribution function with a finite size plateau in velocity space. Then we vary the size of the trapping region, i.e. we consider different values for the electric potential perturbation amplitude, in order to analyze the effect on the oscillation frequency of the wave.

The velocity distribution of particles is chosen to be of the form:

\[
\tilde{f}_0(v_z) = f_M(v_z) - [f_M(v_z) - f_M(v_\phi)] \{1 + [2(v_z - v_\phi)/\Delta v_{\text{trap}}]^{20}\}^{-1}
\]  

\[ \alpha = 0.1 \]

\[ \alpha = 0.4 \]

\[ \alpha = 0.6 \]

\[ \alpha = 1.0 \]

FIG. 6: The density eigen function with a radial temperature profile.
where $f_M$ indicates a Maxwellian function and $\Delta v_{\text{trap}} = 2\sqrt{2e\phi/m}$ ($\phi$ is the amplitude of the potential perturbation),

The results for three different values of the perturbation amplitude are displayed in Fig. 7; the solid line represents the Maxwellian case; the dashed and the dot-dashed lines indicate the solutions corresponding to different values of the wave amplitude. It is easy to see from the figure that the oscillation frequency is shifted upward increasing the potential amplitude $\phi$ even though the effect is very small in correspondence of the value of the wavenumber for the electron trap, $k_z = \pi/L_z \simeq 0.012\lambda_D^{-1}$, indicated by the vertical line in the top and in the bottom plots The bottom plot is a zoom in the region around $k_z \simeq 0.012\lambda_D^{-1}$.

II. 1D EULERIAN VLASOV SIMULATIONS

The numerical results for the electron acoustic waves in finite plasmas presented in this section have been obtained neglecting the radial dependence in the Poisson equation (2). We numerically solve the Vlasov-Poisson equations in the one-dimensional phase space $(z, v_z)$,
taking into account that the electrons trapped in the Penning-Malmberg device bounce back and forth, being reflected by the confining electric potential at the two ends of the trap ($z = 0$ and $z = L_z$).

We can successfully model the specular reflection, considering a numerical box of length $L_{max} = 2L_z$, and imposing boundary conditions in $z$ at the ends of the numerical domain ($z = 0$ and $z = 2L_z$), provided the electric potential perturbation remains an even function with respect to the point $z = L_z$ all along the simulation (obviously the electric field must be an odd function).

The numerical integration of the Vlasov equation has been performed using the splitting method in electrostatic approximation[3], coupled with a finite difference upwind scheme [4]. For convenience, we scale time by the inverse plasma frequency $\omega_p^{-1}$, where $\omega_p = \sqrt{4\pi ne^2/m}$ and $n$ is the electron density. Length is scaled by the Debye length $\lambda_D = \bar{v}/\omega_p$. With these choices, velocity is scaled by the electron thermal velocity $\lambda_D\omega_p = \bar{v}$ and electric field by $\sqrt{4\pi nm\bar{v}^2}$. Since periodic boundary conditions are imposed in the physical space, a standard Fast Fourier routine is used to numerically integrate the Poisson equation. The initial distribution function is a Maxwellian in the velocity space, over which a noise modulation in the physical space with amplitude $A_{noise} = 10^{-4}$ is superposed

$$f(z, v_z, t = 0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v_z^2}{2}} \left[ 1 + A_{noise} \sum_{j=0}^{J_{max}} \cos(k_j z) \right]$$

where $k_j = jk_0$ and $k_0 = 2\pi/L_{max}$ represents the fundamental wave number in the simulation box.

The simulation domain in the phase space is given by $D = [0, L_{max}] \times [-v_{zmax}, v_{zmax}]$, where $L_{max} = 2\pi/k_0$ and $v_{zmax} = 6$. Outside the velocity simulation interval the distribution function is put equal to zero. Typically a simulation is performed using $N_x = 512$ grid points in the physical space and $N_v = 2400$ grid points in the velocity space. The time step $\Delta t \simeq 0.005 - 0.001$ has been chosen in such a way that Courant-Friedrichs-Levy condition (CFL) is satisfied. An energy conservation equation has been used to control numerical accuracy. The total energy variations remain always $10^{-2}$ times smaller with respect to typical electric and kinetic energy fluctuations, all along the simulation.

To successfully excite the electron acoustic wave, we follow the method described by Valentini, O’Neil and Dubin in Ref. [5]. We drive the plasma, using an external driver...
FIG. 8: Time evolution of the electric field spectral component $n = 2$ and $n = 1$.

electric field (a standing wave in the simulation domain), taken to be of the form

\[
E_D(x, t) = E_{D}^{\text{max}} \left[ 1 + \left( \frac{t - \tau}{\Delta \tau} \right)^\beta \right]^{-1} \times \\
\left[ \sin(nk_0x - \omega t) + \sin(nk_0x + \omega t) \right]
\]

(16)

where $E_{D}^{\text{max}} = 0.01$, $\tau = 1200$, $\Delta \tau = 600$, $\beta = 10$, and $n = 2$ (we drive the mode $n = 2$). It is important to note that we choose an electric perturbation of the form of a sine wave and the phase of this wave must not change during the simulation. In particular, the electric field must be zero at the ends of the numerical trap ($z = 0$ and $z = L_z$) all along the simulation. An abrupt turn on (or off) of the driver field would excite Langmuir waves as well as EAWs, complicating the analysis. Thus, the driver is turned on and off adiabatically. The driver amplitude is near $E_{D}^{\text{max}}$ (within a factor of two) for a time $t_{\text{off}} - t_{\text{on}} \simeq 1200$, and is near zero again by $t_{\text{off}} \simeq 2200$. The time dynamics of the system is followed up to $t_{\text{max}} = 4000$. We choose a simulation domain of total length $L_{\text{max}} = 2L_z = 40$, but all the results will be presented for $0 < z < L_z = 20$, that is the actual length of the plasma confined in
the trap. In these conditions, the resonant phase velocity for the electron acoustic waves is $v_\phi = \omega / (n k_0) \simeq 1.70$, as showed in [5].

The length of the numerical box is quite different from the real value of the plasma length in the electron trap, which is $L_z = 258.34 \Rightarrow k_z \simeq 0.012$ (in units of inverse Debye length); however, using the real experimental parameters, the oscillation frequency of the excited electron acoustic wave (the mode $n = 2$) would be of the order of $\omega \simeq 2 \bar{v}(k_z) \simeq 0.024$. Then, the oscillation period of the wave would be very large in unit of the inverse plasma frequency $T \simeq 260$, and consequently the time dynamics of the system would be too slow to be investigated numerically, using the Vlasov code described above.

The time evolution of the electric field spectral component $n = 2$ and $n = 1$ is displayed in Fig. 8. The mode $n=2$ is excited during the driving process; a big plasma response is observed around $t = 1000$, and the perturbation amplitude grows while the driver pumps energy into the system, then the electric field reaches a constant saturation value at $t \simeq 1800$ and goes on oscillating around the saturation value until the driver is turned off at $t = 2200$. Then, the decay instability, already described in Ref. [5] through a Particle in Cell
FIG. 10: Contour plot of the distribution function in phase space, for $t > 2200$. 
simulation, transfers energy from the mode $n = 2$ to the mode $n = 1$ (i.e. to the largest wavelength that fits in the simulation box). The dashed line in the bottom graph in Fig. 8 represents the time $t = t_{off}$; it is easily seen that the mode $n = 1$ starts growing for $t > t_{off}$, then saturates and rings at nearly constant amplitude, for many wave cycles. Obviously, the phase of the noise determines the phase of the daughter wave during the decay instability. It is worth noting that the phase of the noise on the initial distribution function has been chosen in such a way that it produces a superposition of small amplitude sine waves in the electric field, so that we keep the electric field zero at the ends of the numerical trap, also during the evolution of the daughter wave.

As a check for the validity of our model, in Fig. 9 we show the time evolution of the electric field evaluated at the points $z = 0$ and $z = L_z$ in the simulation domain (that corresponds to the ends of the electron trap). As it can easily be seen from the figure, the value of the electric field at $z = 0$ and $z = L_z$ remains smaller than $10^{-10}$ until the end of the simulation.

Finally, the decay instability has been analyzed looking at the contour plot of the electron distribution function in phase space. In Fig. 10, the level lines of the distribution are shown for several times, right after the driver is turned off. At the beginning, two vortex structures are visible in phase space, the first moving with positive velocity $v_z = v_\phi \simeq 1.7$ along the $z$ axis and the other one moving with $v_z = -v_\phi$. Each of them bounces back and forth in the trap, this corresponding to a circular motion in phase space, as it is clear from the figure. During the decay instability the two vortex merge (see Ref. [5]), and finally one big vortex is present in the trap that still bounces back and forth, being reflected at the two ends.