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The Phonon Wake Behind a Charge Moving Relative to a 2D Plasma Crystal

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Abstract

In a recent experiment a wake was created in a 2D lattice of charged dust grains by a charge moving parallel to the lattice plane. Multiple "Mach cones" were observed in the wake. This paper describes a linear theory of the phonon wake caused by a charge moving relative to a crystalline lattice. The theory predicts multiple structures in the wake that match those observed in the experiments. These structures are caused by constructive interference of compressional phonons excited by the moving charge, combined with the strongly dispersive nature of these phonons. The theoretical wake is matched to an experimental wake in order to determine the plasma Debye length and the charge on the dust grains.

1. Introduction

Recently the wake created in a crystal lattice by a moving charge was observed and measured [1,2]. The crystal consisted of a 2D triangular lattice of charged dust grains with large spacing, $a = 256$ microns. The grains were levitated against the force of gravity in the sheath of an rf plasma discharge. A charge moving parallel to the crystal plane with nearly constant speed U of only a few cm/s perturbed the positions of the dust grains, creating a wake in the lattice that could be imaged with a digital camera.

The wake had some expected features as well as several unexplained structures. As expected, a "Mach cone" was observed, in which a perturbation was concentrated in a cone with opening angle β, and had an angle of propagation $\theta = \theta_0$ that obeyed the usual Mach conditions,

$$\begin{aligned} \beta &= \theta_0 \,, \\ c(0) &= U \sin\theta_0 \,, \end{aligned} \tag{1}$$

(where $c(0)$ is the sound speed of long-wavelength phonons). The angles θ and β are defined in Fig. 1. [The angle of propagation for a wave with wavenumber $\boldsymbol{k}$ is $\theta = \tan^{-1}(k_y/k_x)$; the angle β is defined for a given displacement (x_0, y_0) from the moving charge as $\beta = \tan^{-1}(x_0/(-)y_0)$.] Equation (1) merely prescribes that surfaces of constant phase keep pace with the moving charge, so that the driven wave can be resonant with the charge. However, several other "Mach cones" with different (smaller) opening angles also appeared in both experiments and simulations, and these structures were unexplained.

This paper briefly outlines a general linear theory for the wake induced by a charge moving at constant velocity with respect to a 2D crystalline lattice. The theory predicts multiple structures in the wake that can be matched to those observed in the experiments and simulations, by choosing values of the Debye length λ of the background plasma and the charge Q on each dust grain. In fact, we show that the spatial form of the observed wake can be used to determine these two quantities.

The multiple wake structures are a consequence of the strongly-dispersive nature of compressional phonons (sound waves) in a 2D lattice. The excited waves satisfy the Mach condition, $c = U \sin\theta$ but $c = c(\boldsymbol{k})$ so different excited waves travel at different propagation angles, $\theta = \theta(\boldsymbol{k})$. Phase mixing of the various excited waves causes constructive and destructive interference. As a result, along a line defined by some given opening angle β we will show that specific wavenumbers $\boldsymbol{k} = \boldsymbol{k}_0(\beta)$ are dominant, and in general the propagation angle θ for these wavenumbers is not equal to β. These wavenumbers form the observed multiple wakes.

Such structures do not occur in the single Mach cone shock wave surrounding a particle in air that moves faster than the speed of sound. This is because air is much less dispersive than a 2D lattice, so Eq. (1) is nearly correct for all significant wavenumbers. On the other hand, many other media have strong dispersion and therefore also exhibit multiple wake structures. Probably the best known example is the so-called "Kelvin wedge" that forms behind a ship moving in deep water, caused by the strong linear dispersion of deep water surface waves [3].

In a 2D dusty-plasma crystal, the structure of the wake depends on the speed of the moving charge, U, compared to the phase speed of long wavelength phonons, $c(0)$, and the ratio of the Debye length λ to the interparticle spacing a. Two cases, $U/c(0) = 0.8$ and $U/c(0) = 1.8$, are shown in Fig. 2, assuming $\lambda = a$. The figure displays the positions of wave crests in the wake. The theory behind these figures is the subject of the next section.

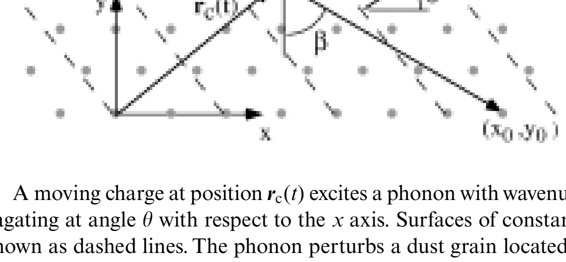

Fig. 1. A moving charge at position $\boldsymbol{r}_c(t)$ excites a phonon with wavenumber $\boldsymbol{k}$, propagating at angle θ with respect to the x axis. Surfaces of constant phase are shown as dashed lines. The phonon perturbs a dust grain located at position (x_0, y_0), measured with respect to the moving charge.

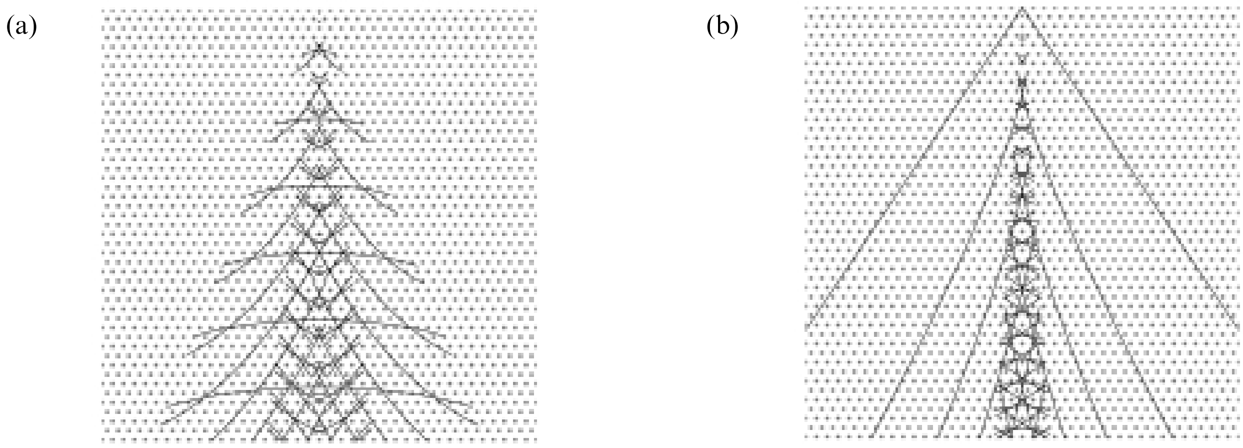

Fig. 2. Structure of the wake behind a charge moving in the positive y direction (up in the figures), assuming $\lambda/a = 1$. Solid lines display wave crests. Gray dots are the positions of charges in the triangular lattice, with spacing shown to scale with respect to the wake. (a) $U/c(0) = 0.8$. b) $U/c(0) = 1.8$.

2. Theory

In this section we present a brief overview of the theory of the wake behind a moving charge. The full theory may be found in Ref. [4]. We consider an infinite lattice of identical charged dust grains, with charge Q and mass m, confined to the $x-y$ plane. The grains interact with one another via a Yukawa potential $\phi(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$, where

$$\phi(r) = Q^2 \frac{\mathrm{e}^{-r/\lambda}}{r} , \tag{3}$$

and where λ is the Debye length of the background plasma. This is a good approximation for particles suspended at the same height in the plasma sheath [5].

A moving charge below the plane, with projected (x, y) position $\boldsymbol{r}_\mathrm{c}(t) = x_\mathrm{c}\hat{\mathbf{x}} + Ut\,\hat{\mathbf{y}}$ at time t, creates a force $-\nabla\Phi(\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{c}(t))$ on the ith dust grain, at position $\boldsymbol{r}_i$. The potential $\Phi(\mathbf{r})$ is not completely understood, since the moving charge is at a different height in the sheath than the dust grains, and various plasma effects, such as the flux of ions through the sheath, can affect the interparticle force. Some research has pointed to an attractive interaction between grains at different heights [6–8]; others have observed grain repulsion in some circumstances [5,9]. In what follows we will leave Φ an unknown function.

Assuming that the perturbed position $\delta\boldsymbol{r}_i$ is small, $\delta\boldsymbol{r}_i$ satisfies the following linearized equation of motion:

$$m\delta\ddot{\boldsymbol{r}}_i = -\nabla\Phi(\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{c}(t)) - \sum_{j\neq i} \frac{\partial^2\phi}{\partial\boldsymbol{r}_i\partial\boldsymbol{r}_i}(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot[\delta\boldsymbol{r}_i - \delta\boldsymbol{r}_j] - m\nu\,\delta\dot{\boldsymbol{r}}_i , \tag{2}$$

where ν is a phenomenological damping rate, caused by collisions of the grains with neutral gas. This linear equation can be solved for the driven response of the dust grains to the force of the moving charge by introducing phonon coordinates and Fourier transforming. The result of this analysis is

$$\delta\boldsymbol{r}_i(t) = -\sum_\alpha \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \frac{\mathrm{i}\hat{\Phi}(\boldsymbol{k})\boldsymbol{k}\cdot\boldsymbol{e}_\alpha(\boldsymbol{k})\boldsymbol{e}_\alpha(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(x_0,y_0)}}{\omega_\alpha^2(\boldsymbol{k}) - k_y^2U^2 - \mathrm{i}\nu k_y U} , \tag{3}$$

where $\hat{\Phi}(\boldsymbol{k}) = \int \mathrm{d}^2\boldsymbol{r}\,\Phi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ is the Fourier integral of $\Phi(\boldsymbol{r})$, $(x_0, y_0) \equiv \boldsymbol{r}_i - \boldsymbol{r}_\mathrm{c}(t)$ are coordinates in a frame that moves with the moving charge, and $\omega_\alpha(\boldsymbol{k})$ and $\boldsymbol{e}_\alpha(\boldsymbol{k})$ are the frequency and polarization unit vector of a phonon with wavenumber $\boldsymbol{k}$. In the wavenumber integral in Eq. (3), ω_α and $\boldsymbol{e}_\alpha$ are periodic functions of $\boldsymbol{k}$, repeating in each Brillouin zone of the 2D triangular lattice. The subscript α denotes the type of phonon. For a 2D lattice, there are 2 types of phonons, termed compressional and shear (for $ka \ll 1$, $\boldsymbol{e}(\boldsymbol{k}) \parallel$ or $\perp \boldsymbol{k}$, respectively). The phase velocity of the compressional phonons is larger than that of the shear branch [1,10]. The wake is therefore dominated by this branch, and we will concentrate on the compressional phonons in the remainder of the paper.

The wavenumber integral in Eq. (3) is dominated by the pole, at $\omega_\alpha^2 = k_y^2U^2 + \mathrm{i}\nu k_y U$. Dropping the imaginary part, we denote $k_{y_0}(k_x)$ as the solution to the resonance condition

$$\omega_\alpha^2(k_x, k_{y_0}) = k_{y_0}^2 U^2 . \tag{4}$$

Note that this resonance condition is just the Mach condition $c \equiv \omega/k = U\sin\theta$, where $\theta = \tan^{-1}(k_y/k_x)$ is the angle of propagation. Keeping only the contribution from the compressional mode, and dropping the polarization subscript α, the k_x and k_y integrals can be evaluated approximately by keeping only the contribution from the pole , and then using the method of stationary phase. The result is

$$\delta\boldsymbol{v}_i = \mathrm{Re}\,\frac{-\mathrm{i}\hat{\Phi}(\boldsymbol{k}_0)}{m\sqrt{2\pi|y_0u(\boldsymbol{k}_0)|}} \frac{\boldsymbol{k}_0\cdot\boldsymbol{e}(\boldsymbol{k}_0)\boldsymbol{e}(\boldsymbol{k}_0)}{U - v_{g_y}(\boldsymbol{k}_0)}\, H\!\left(\frac{y_0}{U - v_{g_y}(\boldsymbol{k}_0)}\right) \times \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0\cdot(x_0,y_0)+(\mathrm{i}\pi/4)\mathrm{sgn}[y_0u(\boldsymbol{k}_0)]+(1/2)(\nu y_0/(U-v_{g_y}(\boldsymbol{k}_0)))} \tag{5}$$

where

$$u(\boldsymbol{k}_0) \equiv \frac{\partial}{\partial k_x}\left(\frac{v_{g_x}}{U - v_{g_y}}\right)\Bigg|_{\boldsymbol{k}=\boldsymbol{k}_0} . \tag{6}$$

Here, $H(x) = [1 - \mathrm{sgn}(x)]/2$ is a Heavyside step function, $\boldsymbol{k}_0 = (k_{x0}, k_{y0})$ and k_{x0} and k_{y0} are wavenumbers found by simultaneous solution of the resonance condition, Eq. (4), and stationary phase condition,

$$\tan\beta \equiv \frac{x_0}{-y_0} = \frac{v_{g_x}}{U - v_{g_y}}\Bigg|_{\boldsymbol{k}=\boldsymbol{k}_0} , \tag{7}$$

where v_{gx} and v_{gy} are the x and y components of the group velocity $\partial\omega/\partial\boldsymbol{k}$. Note that $\boldsymbol{k}_0 = \boldsymbol{k}_0(\beta)$ through Eq. (7), and that Eq. (5) is valid only where $|y_0 u(\boldsymbol{k}_0)| \gg a^2$.

Equation (5) describes a trailing wake ($y_0 < 0$) consisting of oscillations that decay like $e^{-\nu|y_0/2(U-v_{gy})|}/\sqrt{|y_0|}$, with distance $|y_0|$ behind the moving charge. The oscillations have a spatially-varying wavenumber $\boldsymbol{k}_0$ that depends on opening angle β through Eq. (7).

For $U < c(0)$, these oscillations create a wake that is similar to the wake behind a ship [3]. At larger opening angles there are two solutions to Eq. (7) for $\boldsymbol{k}_0(\beta)$, corresponding to a transverse wake with large θ and a lateral wake with smaller θ. These two wakes are superimposed on one another. Assuming that $\hat{\Phi}(\boldsymbol{k})$ is real for all $\boldsymbol{k}$, the oscillations are proportional to $\sin[\boldsymbol{k}_0 \cdot (x_0, y_0) + (\pi/4)\mathrm{sgn}(y_0 u(\boldsymbol{k}_0))]$. The peaks (maxima) in this function are shown as solid lines in Fig. 2a, for the case $\nu = 0$ (no damping), $U/c(0) = 0.8$, and $\lambda/a = 1$ in the Yukawa potential.

The complex set of criss-crossing extrema directly behind the moving charge are superimposed on the slower-varying lateral and transverse wakes. They are a result of the large-wavenumber umklapp phonon solutions for $k_{x_0}(\beta)$. (Umklapp phonons are phonons from beyond the first Brillouin zone of the reciprocal lattice.) The extent to which these solutions actually affect the wake depends on the magnitude of $\hat{\Phi}(\boldsymbol{k})$ at large $\boldsymbol{k}$. For example, if $\hat{\Phi}(\boldsymbol{k})$ were zero for wavenumbers beyond the first Brillouin zone, only the lateral wake and the Mach cone would appear.

For $U > c(0)$, the transverse wake disappears and a new solution of Eq. (7) appears, consisting of a cone emanating from the moving charge, with opening angle β determined by Eq. (1). This "linear Mach cone," along with the crests in the lateral wake and the umklapp wake, is displayed in Fig. 2b for $U/c(0) = 1.8$ and $\lambda/a = 1$.

Note that the wavefronts in the lateral wake curve outward until they are parallel to the Mach cone at large distances. However, if finite damping ν is added to the solution, these wavefronts decay exponentially before they achieve the same opening angle as the Mach cone. The experiments and simulations, which had finite damping, also observed that the secondary wavefronts had smaller opening angles than the Mach cone [1,2].

In Fig. 3 we have superimposed an experimental image of "multiple Mach cones" on the theoretical wake. This could be done because the spacing between dust grains was measured to be 256 μm, and a distance scale was included with the experimental image of the wake [1]. The Mach number, 1.8, was determined by measuring the opening angle of the first cone in the experimental image, and then using Eq. (1). One can see that both cones in the experimental image fit the theory well, provided that the Debye length is chosen properly.

The wavelength of the wake depends on the ratio of the Debye length to the interparticle spacing implicitly through Eq. (7), and the wavelength increases as λ/a increases. This can be seen by comparing Figs. 2b and 3, which are drawn on the same scale and differ only by the value of λ/a. Fig. 2b shows only wave crests, while Fig. 3 shows both crests and troughs. In order to fit to the experimental image one must take care to note that in the second experimental "Mach cone," the wake velocity is reversed in direction compared to the first cone; the second cone is actually a trough in the wake, not a crest. A value of $\lambda/a = 2$ matches the experimental results for both "Mach cones" quite well, as shown in Fig. 3. This value of λ/a is considerably larger than the value which was measured in the experiment using a probe. However, it may be that the plasma density is depressed in the region of the dust lattice plane [1], and in this case $\lambda/a = 2$ may not be not out of the question.

At present it is not known why only one crest and trough were observed in the experiment. It may be that finite damping reduced the magnitude of the other crests and troughs below the sensitivity of the measurement. Several other crests and troughs were observed in simulations [1].

We can use the form of the wake to determine the charge Q on the dust grains. The charge is determined by the numbers $\lambda/a = 2$ and $U/c(0) = 1.8$, together with other measured quantities, the mass of the grains, $m = 5.6 \times 10^{-10}$ g, the interparticle spacing $a = 256$ microns, and the speed of the moving charge, $U = 4$ cm/s. Since $U/c(0) = 1.8$, $U = 4$ cm/s implies $c(0) = 2.2$ cm/s. Now, for $\lambda/a = 2$, the speed of long wavelength compressional phonons in a Yukawa lattice is $c(0) = 3.6(Q^2/ma)^{1/2}$ [11], and therefore $|Q| = 4900e$.

3. Discussion

We have shown that the wake of compressional phonons excited in a 2D crystal lattice by a moving charge has a structure that can be matched to experimental images of "multiple Mach cones" in a dust plasma crystal by choosing appropriate values of the plasma Debye length and the charge on the dust grains. For one experiment, a comparison to the theory determined the plasma Debye length and the charge on the dust grains to be $\lambda \simeq 500\,\mu$m and $|Q| \simeq 4900e$ respectively.

When $U < c(0)$, the first linear Mach cone is predicted to disappear, to be replaced by a transverse wake, similar to that behind a ship (Fig. 2a). This type of wake has not yet been observed.

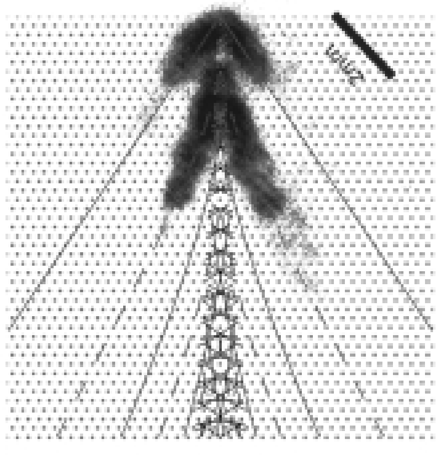

Fig. 3. Wake crests (solid lines) and troughs (dashed lines) for $U/c(0) = 1.8$ and $\lambda/a = 2$. Overlaid on the theory is an experimental density plot of the perturbed speed of dust particles in the wake, taken from Ref. [1]. Darker colors correspond to higher speeds. The bar labelled 2 mm provides a distance scale for comparison to the theoretical wake.

The structure of the wake depends on the interaction between the dust grains and the moving charge. The form of this interaction is a subject of current research in several groups [5–9]. Although we did not discuss this point here, our theory analysis of the wakes observed in Refs. [1] and [2] shows that the force between the moving charge and the dust grains cannot be purely attractive. A significant repulsive component to the force must also be present. This is discussed in more detail in Ref. [4].

Acknowledgements

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